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Serge Piperno, Loula Fatima Fezoui

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Abstract: A Discontinuous Galerkin method is applied here to the numerical solution of the time-domain Maxwell's equations on unstructured meshes. The method relies on the choice of a local basis of functions, a centered mean approximation for the surface integrals and a second-order leap-frog scheme for advancing in time. The method is proved to be stable for a large class of basis functions and a discrete analog of the electromagnetic energy is also conserved.

Key-words: electromagnetism, finite volume methods, discontinuous Galerkin, centered fluxes, leap-frog time scheme, L^2 stability, unstructured meshes, absorbing boundary condition

* CERMICS, INRIA, BP93, F-06902 Sophia-Antipolis Cedex, Serge.Piperno@sophia.inria.fr

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

Un schéma en volumes-finis non-structurés centré de type Glerkin discontinu pour la résolution des équations de Maxwell tridimensionnelles en milieu hétérogène

Résumé : Nous présentons une nouvelle méthode de Galerkin Discontinue appliquée à la résolution numérique des équations de Maxwell en maillages non structurés. La méthode repose sur le choix d'une base locale de fonctions, une formulation centrée pour approcher les intégrales de surface et un schéma saute-mouton d'ordre deux. Nous montrons que la méthode conserve une enérgie discrète et une condition suffisante de stabilité est démontrée pour une large classe de fonctions de base.

Mots-clés : électromagnétisme, volumes finis, Galerkin discontinu, flux centrés, schéma saute-mouton, stabilité L^2, maillage non structuré, condition limite absorbante

1 Introduction

The DG (Discontinuous Galerkin) methods enjoy a renewed favor nowadays and are now used in many and various applications [1] as people discover the abilities of these methods to handle complicated geometries and meshes, to achieve a high order of accuracy by simply choosing suitable basis functions, to allow a wide range for time integration schemes and last but not least to remain highly parallelizable at the end. Obviously this has a cost in time and memory on computers especially if we don't take care of the way the surface integrals are evaluated and of the time scheme used. So we have already developed a DG method for the Maxwell equations on triangular meshes using a Gauss quadrature formula and a three step Runge-Kutta scheme which lead to a very costly scheme hardly extensible to the three-dimensional case [5].

We present here a new formulation considering the specific characters of the system to be solved, namely the time domain Maxwell equations. Since we want to preserve the conservation of the discrete analog of the electromagnetic energy, we choose a leap-frog scheme for the time integration as it is the case for the Yee scheme which remains the most used in CEM although its severe restriction to Cartesian grids. We also decide to give up the Gauss quadrature formula whose complexity and cost grow with the accuracy and the space dimension and we simply evaluate the surface integrals via a centered mean of tangent fields on either side of the surface, the fields being projected on the local basis functions.

We dress the outline of the method in the general case in the first section of this paper, then we analyze the stability of the resulting scheme and the conservation of a discrete energy in the two following sections. A sufficient stability condition is proved when the system is provided with one of the two classical boundary conditions, a perfect metallic condition on a material surface and an absorbing one on the artificial boundary delimiting the numerical domain. The fourth section deals with the particular case of tetrahedral meshes using the local $\mathbb{P}_1$ (piecewise affine) basis functions. The resulting scheme was implemented and some numerical results are presented and compared with the exact solutions. Let us note that using piecewise constant functions as a basis ($\mathbb{P}_0$), will result in a centered finite volume scheme which was already presented and studied [3, 4].

We consider in this paper Maxwell equations in three space dimensions for heterogeneous anisotropic linear media with no source. The electric permittivity tensor $\bar{\bar{\varepsilon}}(x)$ and the magnetic permeability tensor $\bar{\bar{\mu}}(x)$ are varying in space and both symmetric positive definite. The electric field $\vec{E} = {}^t(E_x, E_y, E_z)$ and the magnetic field $\vec{H} = {}^t(H_x, H_y, H_z)$ verify

$$\begin{cases} \bar{\bar{\varepsilon}}\dfrac{\partial \vec{E}}{\partial t} = \vec{\mathrm{rot}}\vec{H}, \\ \bar{\bar{\mu}}\dfrac{\partial \vec{H}}{\partial t} = -\vec{\mathrm{rot}}\vec{E}. \end{cases} \tag{1}$$

These equations are set and solved on a bounded polyhedral domain Ω of $\mathbb{R}^3$. Everywhere on the domain boundary $\partial\Omega$ (of unitary outwards normal $\vec{n}$), a boundary condition is set which is either metallic ($\vec{n} \times \vec{E} = \vec{0}$, on $\partial\Omega_m$) or absorbing ($\vec{n} \times \vec{E} = -c\mu\ \vec{n} \times \left(\vec{n} \times \vec{H}\right)$, on $\partial\Omega_a$, where we assume the medium is isotropic, i.e. $\bar{\bar{\varepsilon}} = \epsilon \mathbb{I}_3$ $\bar{\bar{\mu}} = \mu \mathbb{I}_3$ and the local light speed c is given by $\epsilon\mu c^2 = 1$). Examples of such frameworks are given on Figure 1).

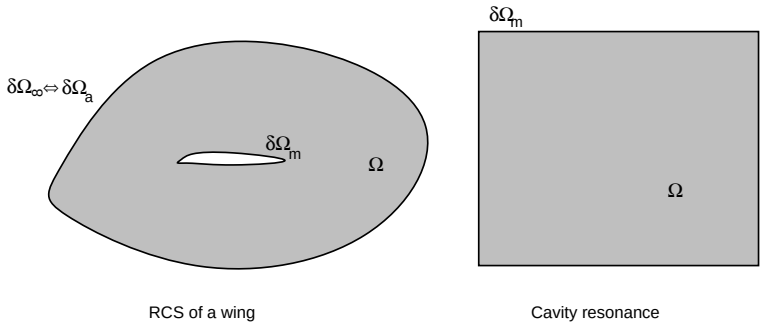

Figure 1: Domains Ω and corresponding boundaries.

2 The new Discontinuous Galerkin FVTD method

2.1 Introduction

We assume we dispose of a partition of the polyhedral domain Ω into a finite number of polyhedra (each one having a finite number of faces). For each polyhedron $\mathcal{T}_i$, called "finite volume" or "cell", V_i denotes its volume, and $\bar{\bar{\varepsilon}}_i$ and $\bar{\bar{\mu}}_i$ are respectively the local electric permittivity and magnetic permeability tensors of the medium, which could be varying inside the cell $\mathcal{T}_i$. We call interface between two finite volumes their intersection, whenever it is a polyhedral surface. For each internal interface $a_{ik} = \mathcal{T}_i \bigcap \mathcal{T}_k$, we denote by $\vec{n}_{ik}$ the integral over the interface of the unitary normal, oriented from $\mathcal{T}_i$ towards $\mathcal{T}_k$. The same definitions are extended to boundary interfaces (in the intersection of the domain boundary $\partial\Omega_m \bigcup \partial\Omega_a$ with a finite volume), the index k corresponding to a fictitious cell outside the domain. We denote by $\vec{\tilde{n}}_{ik} = {}^t(\tilde{n}_{ikx}, \tilde{n}_{iky}, \tilde{n}_{ikz})$ the normalized normals $\vec{\tilde{n}}_{ik} = \vec{n}_{ik}/\|\vec{n}_{ik}\|$.

Finally, we denote by $\mathcal{V}_i$ the set of indices of the neighboring finite volumes of the finite volume $\mathcal{T}_i$ (having an interface in common). We also define the perimeter P_i of $\mathcal{T}_i$ by $P_i = \sum_{k \in \mathcal{V}_i} \|\vec{n}_{ik}\|$. We have the following geometrical property for all finite volumes,

$$\sum_{k \in \mathcal{V}_i} \vec{n}_{ik} = 0. \tag{2}$$

2.2 The spatial discretization

Inside each finite volume, the numerical unknowns of the method are related to the orthogonal (in the sense of the classical L^2 scalar product) projection of the electric and magnetic fields on a chosen set of vector basis functions $\vec{\varphi}_{ij}, 1 \leq j \leq d_i$, where d_i denotes the number of local scalar degrees of freedom inside the finite volume $\mathcal{T}_i$. The approximation is allowed to be discontinuous across element boundaries.

We now derive the spatial discretization. Taking (1), dot-multiplying by a given basis function $\vec{\varphi}_{ij}$, and integrating over $\mathcal{T}_i$ yields

$$\left\{ \begin{array}{l} \displaystyle\int_{\mathcal{T}_i} {}^t\vec{\varphi}_{ij}\bar{\bar{\varepsilon}}_i \frac{\partial \vec{E}}{\partial t} = \int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{H}.\vec{\varphi}_{ij}, \\ \displaystyle\int_{\mathcal{T}_i} {}^t\vec{\varphi}_{ij}\bar{\bar{\mu}}_i \frac{\partial \vec{H}}{\partial t} = -\int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{E}.\vec{\varphi}_{ij}. \end{array} \right.$$

Using the identity $\vec{\mathrm{rot}}\vec{X}.\vec{\psi} = \vec{\mathrm{rot}}\vec{\psi}.\vec{X} - \mathrm{div}(\vec{\psi} \times \vec{X})$, we get

$$\left\{ \begin{array}{l} \displaystyle\int_{\mathcal{T}_i} {}^t\vec{\varphi}_{ij}\bar{\bar{\varepsilon}}_i \frac{\partial \vec{E}}{\partial t} = \int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{\varphi}_{ij}.\vec{H} - \int_{\partial\mathcal{T}_i} (\vec{\varphi}_{ij} \times \vec{H}).\vec{\vec{n}}, \\ \displaystyle\int_{\mathcal{T}_i} {}^t\vec{\varphi}_{ij}\bar{\bar{\mu}}_i \frac{\partial \vec{H}}{\partial t} = -\int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{\varphi}_{ij}.\vec{E} + \int_{\partial\mathcal{T}_i} (\vec{\varphi}_{ij} \times \vec{E}).\vec{\vec{n}}. \end{array} \right. \tag{3}$$

If we denote by $\vec{\mathbf{E}}_i$ and $\vec{\mathbf{H}}_i$ respectively the canonical L^2-orthogonal projections of the fields $\vec{E}$ and $\vec{H}$ on $Span(\vec{\varphi}_{ij},\ 1 \leq j \leq d_i)$ inside the finite volume $\mathcal{T}_i$, verifying the property

$$\forall \vec{\varphi} \in Span(\vec{\varphi}_{ij},\ 1 \leq j \leq d_i), \quad \int_{\mathcal{T}_i} \vec{\mathbf{E}}_i.\vec{\varphi} = \int_{\mathcal{T}_i} \vec{E}.\vec{\varphi}, \quad \int_{\mathcal{T}_i} \vec{\mathbf{H}}_i.\vec{\varphi} = \int_{\mathcal{T}_i} \vec{H}.\vec{\varphi},$$

then, in equations (3), $\vec{\mathbf{E}}_i$ and $\vec{\mathbf{H}}_i$ (and their time-derivative) can be directly used to evaluate volume integrals. For boundary integrals, since no continuity is imposed on the fields, some additional approximations have to be done. We choose here to use completely centered fluxes, i.e.

$$k \in \mathcal{V}_i,\ \forall x \in a_{ik},\ \vec{E}(x) \to \frac{\vec{\mathbf{E}}_i(x) + \vec{\mathbf{E}}_k(x)}{2}, \quad \vec{H}(x) \to \frac{\vec{\mathbf{H}}_i(x) + \vec{\mathbf{H}}_k(x)}{2}.$$

The fields $\vec{\mathbf{E}}_i$ and $\vec{\mathbf{H}}_i$ are then decomposed the following way:

$$\forall x \text{ in } \mathcal{T}_i, \quad \vec{\mathbf{E}}_i(x,t) = \sum_{1 \leq j \leq d_i} E_{ij}(t)\ \vec{\varphi}_{ij}(x), \quad \vec{\mathbf{H}}_i(x,t) = \sum_{1 \leq j \leq d_i} H_{ij}(t)\ \vec{\varphi}_{ij}(x). \tag{4}$$

Inside each control volume, the fields $\vec{\mathbf{E}}_i$ and $\vec{\mathbf{H}}_i$ can now be represented using a chosen number of scalar values E_{il} and H_{il}, for $1 \leq l \leq d_i$. We will now denote by $\mathbf{E}_i$ the column

$(E_{il})_{1\leq l\leq d_i}$. Finally, this leads to

$$\left\{\begin{array}{l} \left(M_i^{\epsilon}\dfrac{\partial \mathbf{E}_i}{\partial t}\right)_j = \displaystyle\int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{\varphi}_{ij}.\vec{\mathbf{H}}_i - \sum_{k\in\mathcal{V}_i}\left(\int_{a_{ik}} (\vec{\varphi}_{ij}\times\frac{\vec{\mathbf{H}}_i+\vec{\mathbf{H}}_k}{2}).\vec{\tilde{n}}_{ik}\right), \\ \left(M_i^{\mu}\dfrac{\partial \mathbf{H}_i}{\partial t}\right)_j = -\displaystyle\int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{\varphi}_{ij}.\vec{\mathbf{E}}_i + \sum_{k\in\mathcal{V}_i}\left(\int_{a_{ik}} (\vec{\varphi}_{ij}\times\frac{\vec{\mathbf{E}}_i+\vec{\mathbf{E}}_k}{2}).\vec{\tilde{n}}_{ik}\right), \end{array}\right. \tag{5}$$

where the j subscripts denote the jth component of vectors, the fields $\vec{\mathbf{E}}_i$ and $\vec{\mathbf{H}}_i$ are given in (4) in functions of scalar degrees of freedom, and M_i^{ϵ} and M_i^{μ} are square matrices of size d_i, given by

$$\begin{array}{rcl} (M_i^{\epsilon})_{jl} & = & \displaystyle\int_{\mathcal{T}_i} {}^t\vec{\varphi}_{ij}\bar{\bar{\varepsilon}}_i\vec{\varphi}_{il},\ 1\leq j,l\leq d_i, \\ (M_i^{\mu})_{jl} & = & \displaystyle\int_{\mathcal{T}_i} {}^t\vec{\varphi}_{ij}\bar{\bar{\mu}}_i\vec{\varphi}_{il},\ 1\leq j,l\leq d_i. \end{array} \tag{6}$$

It is clear that the matrices M_i^{ϵ} and M_i^{μ} are symmetric and definite positive, because the tensors $\bar{\bar{\varepsilon}}_i$ and $\bar{\bar{\mu}}_i$ are symmetric definite positive, and the basis functions $\vec{\varphi}_{ij}$ are assumed linearly independent.

We shall now prove an energy conservation property for the ordinary differential system (5) (the semi-discretized in space Maxwell equations). Let us first define the following electromagnetic energies:

Definition 2.1 *We consider the following electromagnetic energies inside each finite volume and in an arbitrary connected group $\mathcal{G}$ of finite volumes:*

(i) $\forall i,\ \mathbb{E}_i = \frac{1}{2}\displaystyle\int_{\mathcal{T}_i}\left({}^t\vec{\mathbf{E}}_i\bar{\bar{\varepsilon}}_i\vec{\mathbf{E}}_i + {}^t\vec{\mathbf{H}}_i\bar{\bar{\mu}}_i\vec{\mathbf{H}}_i\right) = \frac{1}{2}\left({}^t\mathbf{E}_i M_i^{\epsilon}\mathbf{E}_i + {}^t\mathbf{H}_i M_i^{\mu}\mathbf{H}_i\right),$

(ii) $\mathbb{E}_{\mathcal{G}} = \displaystyle\sum_{i\in\mathcal{G}}\mathbb{E}_i.$

We aim at evaluating the energy $\mathbb{E}_{\mathcal{G}}$ time-derivative. Inside each finite volume, equation (5) yields:

$$\begin{aligned}
\frac{\partial \mathbb{E}_i}{\partial t} &= {}^t\mathbf{E}_i M_i^{\varepsilon} \frac{\partial \mathbf{E}_i}{\partial t} + {}^t\mathbf{H}_i M_i^{\mu} \frac{\partial \mathbf{H}_i}{\partial t} \\
&= \sum_{1\le j\le d_i} E_{ij} \left[\int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{\varphi}_{ij}.\vec{\mathbf{H}}_i - \sum_{k\in\mathcal{V}_i} \left(\int_{a_{ik}} (\vec{\varphi}_{ij} \times \frac{\vec{\mathbf{H}}_i + \vec{\mathbf{H}}_k}{2}).\vec{\tilde{n}}_{ik} \right) \right] \\
&\quad - \sum_{1\le j\le d_i} H_{ij} \left[\int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{\varphi}_{ij}.\vec{\mathbf{E}}_i - \sum_{k\in\mathcal{V}_i} \left(\int_{a_{ik}} (\vec{\varphi}_{ij} \times \frac{\vec{\mathbf{E}}_i + \vec{\mathbf{E}}_k}{2}).\vec{\tilde{n}}_{ik} \right) \right] \\
&= \int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{\mathbf{E}}_i.\vec{\mathbf{H}}_i - \sum_{k\in\mathcal{V}_i} \left(\int_{a_{ik}} (\vec{\mathbf{E}}_i \times \frac{\vec{\mathbf{H}}_i + \vec{\mathbf{H}}_k}{2}).\vec{\tilde{n}}_{ik} \right) \\
&\quad - \int_{\mathcal{T}_i} \vec{\mathrm{rot}}\vec{\mathbf{H}}_i.\vec{\mathbf{E}}_i + \sum_{k\in\mathcal{V}_i} \left(\int_{a_{ik}} (\vec{\mathbf{H}}_i \times \frac{\vec{\mathbf{E}}_i + \vec{\mathbf{E}}_k}{2}).\vec{\tilde{n}}_{ik} \right) \\
&= \frac{1}{2} \int_{\mathcal{T}_i} \left(\vec{\mathrm{rot}}\vec{\mathbf{E}}_i.\vec{\mathbf{H}}_i + \vec{\mathrm{rot}}\vec{\mathbf{H}}_i.\vec{\mathbf{E}}_i \right) - \frac{1}{2} \sum_{k\in\mathcal{V}_i} \left(\int_{a_{ik}} \vec{\mathbf{E}}_i \times \vec{\mathbf{H}}_k.\vec{\tilde{n}}_{ik} \right) \\
&\quad - \frac{1}{2} \int_{\mathcal{T}_i} \left(\vec{\mathrm{rot}}\vec{\mathbf{E}}_i.\vec{\mathbf{H}}_i + \vec{\mathrm{rot}}\vec{\mathbf{H}}_i.\vec{\mathbf{E}}_i \right) - \frac{1}{2} \sum_{k\in\mathcal{V}_i} \left(\int_{a_{ik}} \vec{\mathbf{E}}_k \times \vec{\mathbf{H}}_i.\vec{\tilde{n}}_{ik} \right) \\
&= - \sum_{k\in\mathcal{V}_i} \left(\int_{a_{ik}} \frac{\vec{\mathbf{E}}_i \times \vec{\mathbf{H}}_k + \vec{\mathbf{E}}_k \times \vec{\mathbf{H}}_i}{2}.\vec{\tilde{n}}_{ik} \right).
\end{aligned}$$

In the expression of $\frac{\partial \mathbb{E}_{\mathcal{G}}}{\partial t}$ derived from the definition of $\mathbb{E}_{\mathcal{G}}$, all terms corresponding to interfaces a_{ik} internal to the group $\mathcal{G}$ vanish. Only boundary terms are conserved, and this simply leads to:

$$\frac{\partial \mathbb{E}_{\mathcal{G}}}{\partial t} = - \sum_{\text{faces } a_{ik}}^{\text{boundary}} \left(\int_{a_{ik}} \vec{\tilde{n}}_{ik}.\frac{\vec{\mathbf{E}}_i(x) \times \vec{\mathbf{H}}_k(x) + \vec{\mathbf{E}}_k(x) \times \vec{\mathbf{H}}_i(x)}{2} \right). \tag{7}$$

This expression is a discrete version of Poynting's theorem. We recall here that the electromagnetic energy $\mathcal{E}$ in the continuous case (Maxwell system with no current) is given by $\mathcal{E} = 1/2({}^t\vec{E}\bar{\bar{\varepsilon}}\vec{E} + {}^t\vec{H}\bar{\mu}\vec{H})$, and verifies the following conservation equation: $\frac{\partial \mathcal{E}}{\partial t} + \mathrm{div}\vec{P} = 0$, where $\vec{P}$ is Poynting's vector given by $\vec{P} = \vec{E} \times \vec{H}$. Integrating the conservation equation for $\mathcal{E}$ over any closed volume V with a regular boundary ∂V yields Poynting's theorem:

$$\int_V \frac{\partial \mathcal{E}}{\partial t} dv + \int_{\partial V} \vec{P} \cdot \vec{n}\ ds = 0.$$

For example, for a given metallic cavity, since $\vec{E} \times \vec{n} = 0$ at the boundary, Poynting's theorem yields that the electromagnetic energy is exactly conserved in the cavity.

2.3 Weak treatment of boundary conditions

The metallic and absorbing conditions are dealt with in a weak sense by taking some values for the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$ inside the fictitious finite volume beyond the boundary face. In the two cases, a_{ik} denotes a boundary face between a boundary cell $\mathcal{V}_i$ and its fictitious neighbour $\mathcal{V}_k$. For an absorbing boundary face a_{ik}, the fictitious discrete values will be detailed in the sequel. For a metallic boundary face a_{ik}, we use fictitious discrete values E_{kj} and H_{kj}, for $1 \leq j \leq d_k$ such that

$$\begin{array}{l} \bullet\ \forall x \in a_{ik},\ \vec{\mathbf{H}}_k(x) = \vec{\mathbf{H}}_i(x) \quad \text{(continuity of the magnetic field through } a_{ik}\text{)}, \\ \bullet\ \forall x \in a_{ik},\ \vec{\mathbf{E}}_k(x) = -\vec{\mathbf{E}}_i(x) \quad \text{(i.e. } \vec{\mathbf{E}}_i(x) + \vec{\mathbf{E}}_k(x) = \vec{0}\text{)}. \end{array} \tag{8}$$

If only metallic boundary conditions are used the energy is exactly conserved, as stated by the following lemma.

Lemma 2.1 *For solutions of the semi-discretized Maxwell equations (5) with metallic boundaries only (the values given in (8) are used), the discrete electromagnetic energy defined in Definition (2.1) for the whole finite volume partition $\mathcal{G}$ is* **exactly conserved**, *i.e.* $\frac{\partial \mathbb{E}_{\mathcal{G}}}{\partial t} = 0$.

Proof: This is a direct consequence of (7) where all boundary faces a_{ik} are metallic and verify $\vec{\mathbf{E}}_i(x) + \vec{\mathbf{E}}_k(x) \equiv 0$ and $\vec{\mathbf{H}}_i(x) \equiv \vec{\mathbf{H}}_k(x)$.

2.4 The time discretization

We propose to use a leap-frog time discretization. This kind of time scheme has both advantages to be explicit and to be free of time-dissipation. In the sequel, superscripts refer to time stations and Δt is the fixed time-step. The unknowns related to the electric field are approximated at integer time-stations $t^n = n\Delta t$ and are denoted by E_{ij}^n. The unknowns related to the magnetic field are approximated at half-integer time-stations $t^{n+1/2} = (n + 1/2)\Delta t$ and are denoted by $H_{ij}^{n+1/2}$. All definitions for $\mathbf{E}_i^n$, $\mathbf{H}_i^{n+1/2}$, $\vec{\mathbf{E}}_i^n$, and $\vec{\mathbf{H}}_i^{n+1/2}$ are similarly extended. The time scheme directly derives from equation (5) and can be written:

$$\left(M_i^{\epsilon} \frac{\mathbf{E}_i^{n+1} - \mathbf{E}_i^n}{\Delta t} \right)_j = \int_{\mathcal{T}_i} \vec{\mathrm{rot}} \vec{\varphi}_{ij} . \vec{\mathbf{H}}_i^{n+1/2} - \sum_{k \in \mathcal{V}_i} \int_{a_{ik}} (\vec{\varphi}_{ij} \times \frac{\vec{\mathbf{H}}_i^{n+1/2} + \vec{\mathbf{H}}_k^{n+1/2}}{2}) . \vec{\tilde{n}}_{ik}, \tag{9}$$

$$\left(M_i^{\mu} \frac{\mathbf{H}_i^{n+3/2} - \mathbf{H}_i^{n+1/2}}{\Delta t} \right)_j = - \int_{\mathcal{T}_i} \vec{\mathrm{rot}} \vec{\varphi}_{ij} . \vec{\mathbf{E}}_i^{n+1} + \sum_{k \in \mathcal{V}_i} \int_{a_{ik}} (\vec{\varphi}_{ij} \times \frac{\vec{\mathbf{E}}_i^{n+1} + \vec{\mathbf{E}}_k^{n+1}}{2}) . \vec{\tilde{n}}_{ik}. \tag{10}$$

For the treatment of boundary conditions, the fictitious values of fields are simply deduced from previous expressions (in the time-continuous case). For example, for a metallic boundary face a_{ik}, we use

$$\text{Metallic boundary:} \quad \forall x \in a_{ik},\ \vec{\mathbf{E}}_k^n(x) = -\vec{\mathbf{E}}_i^n(x),\ \vec{\mathbf{H}}_k^{n+1/2}(x) = \vec{\mathbf{H}}_i^{n+1/2}(x). \tag{11}$$

For absorbing boundary conditions, this will de detailed in a specific scheme in section 4.

3 A sufficient stability condition for metallic cavities

We aim at giving and proving a sufficient condition for the L^2-stability of the new Galerkin-Discontinuous scheme (9-10) with only metallic boundary conditions. We use the same kind of energy approach as in [2], where a quadratic form plays the role of a Lyapunov function of the whole set of numerical unknowns.

3.1 A discrete energy

We first propose the following discrete energies, directly derived from Definition 2.1 :

Definition 3.1 *For a complete polyhedral finite volume partition $\mathcal{G}$ of the domain Ω with only metallic boundary conditions ($\partial\Omega = \partial\Omega_m$), we consider the following electromagnetic energies inside each finite volume and in the whole domain:*

(i) $\forall i,\ \mathbb{E}_i^n = \frac{1}{2}\int_{\mathcal{T}_i} \left({}^t\vec{\mathbf{E}}_i^n \bar{\bar{\varepsilon}}_i \vec{\mathbf{E}}_i^n + {}^t\vec{\mathbf{H}}_i^{n-1/2} \bar{\bar{\mu}}_i \vec{\mathbf{H}}_i^{n+1/2} \right) = \frac{1}{2}\left({}^t\mathbf{E}_i^n M_i^\varepsilon \mathbf{E}_i^n + {}^t\mathbf{H}_i^{n-1/2} M_i^\mu \mathbf{H}_i^{n+1/2} \right),$

(ii) $\mathbb{E}^n = \sum_{i\in\mathcal{G}} \mathbb{E}_i^n .$

It is absolutely not obvious why the discrete energy $\mathbb{E}^n$ should be a positive definite quadratic form of all numerical unknowns We notice here that the situation is quite different from the proof of the L^2-stability of the first-order upwind finite-volume scheme of [2], where the energy was obviously a positive definite quadratic form of all unknowns. At the same time, the energy proposed here depends explicitly on the numerical scheme, since it can be only written as a quadratic form of all unknowns $(\mathbf{E}_i^n, \mathbf{H}_i^{n-1/2})$ through the use of the second part of the scheme (9-10) with metallic boundary values (11).

In the following, we shall prove that the proposed energy is **conserved** through a time step and that it is a **positive definite quadratic form** of all unknowns **under a CFL-like condition** on the time-step Δt. This will yield the proof that the scheme (9-10) with metallic boundary values (11) is L^2-stable under a condition on Δt.

3.2 Conservation of the discrete energy

Lemma 3.1 *Using the scheme (9)-(10) for an arbitrary connected group $\mathcal{G}$ of finite volumes, the variation during one time step of the discrete electromagnetic energy inside the group, defined in Definition 3.1 is given by*

$$\mathbb{E}_{\mathcal{G}}^{n+1} = \mathbb{E}_{\mathcal{G}}^n + \Delta t \sum_{faces\ a_{ik}}^{boundary} \left(\int_{a_{ik}} \vec{n}_{ik} . \frac{\vec{\mathbf{E}}_i^{n+1/2}(x) \times \vec{\mathbf{H}}_k^{n+1/2}(x) + \vec{\mathbf{E}}_k^{n+1/2}(x) \times \vec{\mathbf{H}}_i^{n+1/2}(x)}{2} \right),$$

with the convention $\vec{\mathbf{E}}_i^{n+1/2}(x) \equiv (\vec{\mathbf{E}}_i^n(x) + \vec{\mathbf{E}}_i^{n+1}(x))/2.$

Proof: The ordinary differential system (5) can be formally seen as a system of the form

$$\left\{ \begin{array}{l} S\dfrac{X^{n+1}-X^n}{\Delta t} = UY^{n+1/2}, \\ T\dfrac{Y^{n+3/2}-Y^{n+1/2}}{\Delta t} = VX^{n+1}, \end{array} \right.$$

where S and T are squared symmetric definite positive matrices, and U and V are rectangular matrices. The discrete electromagnetic energy inside the group, defined in Definition 3.1 is also equal to $F^n = ({}^tX^nSX^n + {}^tY^{n-1/2}TY^{n+1/2})/2$. It is elementary to prove that

$$F^{n+1} = F^n + \Delta t\ {}^tY^{n+1/2}\left(V + {}^tU\right)X^{n+1/2},$$

with $X^{n+1/2} = (X^n + X^{n+1})/2$, which leads after rewriting to the result of the lemma.

Lemma 3.2 *Using the scheme (9)-(10)-(11), the total discrete electromagnetic energy defined in Definition 3.1 is* **exactly conserved**, *i.e.* $\mathbb{E}^{n+1} = \mathbb{E}^n$.

Proof: This is a direct consequence of Lemma 3.1 for the ordinary differential system (5)-(8), because all boundaries are metallic (the proof is as simple as the one in the time-continuous case).

3.3 Definite positivity of the discrete energy

In order to prove that our scheme is stable, we finally show that the discrete energy $\mathbb{E}^n$, under some stability condition on Δt, is a positive definite quadratic form of the numerical unknowns $\mathbf{H}_i^{n-1/2}$ and $\mathbf{E}_i^n$. This will lead to the stability result of this section. We first need some elementary definitions.

Definition 3.2 *Since the basis functions $\vec{\varphi}_{ij}$, $1 \le j \le d_i$ are linearly independent, and since the tensors $\bar{\bar{\varepsilon}}_i$ and $\bar{\bar{\mu}}_i$ are symmetric positive definite, there exists two positive constants ϵ_i and μ_i such that*

$$\forall \vec{\mathbf{X}} \in Span(\vec{\varphi}_{ij},\ 1 \le j \le d_i),\ \int_{\mathcal{T}_i}{}^t\vec{\mathbf{X}}\bar{\bar{\varepsilon}}_i\vec{\mathbf{X}} \ge \epsilon_i\|\vec{\mathbf{X}}\|^2_{\mathcal{T}_i},\ \int_{\mathcal{T}_i}{}^t\vec{\mathbf{X}}\bar{\bar{\mu}}_i\vec{\mathbf{X}} \ge \mu_i\|\vec{\mathbf{X}}\|^2_{\mathcal{T}_i}. \tag{12}$$

We have denoted by $\|\vec{\mathbf{X}}\|_{\mathcal{T}_i}$ the L^2 norm of the vector field $\vec{X}$ over $\mathcal{T}_i$, i.e. $\|\vec{\mathbf{X}}\|^2_{\mathcal{T}_i} = \int_{\mathcal{T}_i}\|\vec{\mathbf{X}}\|^2$. The same notation will also be used for L^2 norm of vector fields over interfaces a_{ik}.

Definition 3.3 *We also assume some regularity of the basis functions $\vec{\varphi}_{ij}$, $1 \le j \le d_i$. More precisely, we assume that for any finite volume $\mathcal{T}_i$, there exists constants α_i and β_{ik} ($k \in \mathcal{V}_i$) such that*

$$\forall \vec{\mathbf{X}} \in Span(\vec{\varphi}_{ij},\ 1 \le j \le d_i), \qquad \|\vec{rot}\vec{\mathbf{X}}\|_{\mathcal{T}_i} \le \frac{\alpha_i P_i}{V_i}\|\vec{\mathbf{X}}\|_{\mathcal{T}_i}, \tag{13}$$

$$\forall \vec{\mathbf{X}} \in Span(\vec{\varphi}_{ij},\ 1 \le j \le d_i), \qquad \|\vec{\mathbf{X}}\|^2_{a_{ik}} \le \frac{\beta_{ik}\|\vec{n}_{ik}\|}{V_i}\ \|\vec{\mathbf{X}}\|^2_{\mathcal{T}_i}. \tag{14}$$

Lemma 3.3 *Using the scheme (9)-(10)-(11), under assumptions of Definitions 3.2 and 3.3, the local discrete electromagnetic energy $\mathbb{E}_i^n$ defined in Definition 3.1 verifies,*

$$\begin{aligned}\mathbb{E}_i^n \quad \geq \quad & \frac{\epsilon_i}{2}\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}^2 + \frac{\mu_i}{2}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 - \frac{\alpha_i P_i \Delta t}{2V_i}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i} \\ & -\frac{\Delta t}{8}\sum_{k\in\mathcal{V}_i}\left(\frac{\beta_{ik}\|\vec{n}_{ik}\|}{V_i}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 + \frac{\beta_{ki}\|\vec{n}_{ik}\|}{V_k}\|\vec{\mathbf{E}}_k\|_{\mathcal{T}_k}^2\right).\end{aligned}$$

In the above expression, if the face a_{ik} is a metallic boundary face, then we set by convention $\|\vec{\mathbf{E}}_k\|_{\mathcal{T}_k} \equiv \|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}$, $\|\vec{\mathbf{H}}_k\|_{\mathcal{T}_k} \equiv \|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}$, $\beta_{ki} \equiv \beta_{ik}$, $V_k \equiv V_i$, $\epsilon_k \equiv \epsilon_i$, and $\mu_k \equiv \mu_i$.

Proof: We get back to the definition of the discrete energy inside a finite volume $\mathbb{E}_i^n$. We have

$$\begin{aligned}\mathbb{E}_i^n &= \frac{1}{2}{}^t\mathbf{E}_i^n M_i^\epsilon \mathbf{E}_i^n + \frac{1}{2}{}^t\mathbf{H}_i^{n\text{-}1/2} M_i^\mu \mathbf{H}_i^{n+1/2} \\ &= \frac{1}{2}{}^t\mathbf{E}_i^n M_i^\epsilon \mathbf{E}_i^n + \frac{1}{2}{}^t\mathbf{H}_i^{n\text{-}1/2} M_i^\mu \mathbf{H}_i^{n\text{-}1/2} - \frac{\Delta t}{2}\mathbb{X}_i^n, \text{ with} \\ \mathbb{X}_i^n &= \sum_{1\leq j\leq d_i} {}^t H_{ij}^{n\text{-}1/2}\left(\int_{\mathcal{T}_i}\vec{\text{rot}}\vec{\varphi}_{ij}.\vec{\mathbf{E}}_i^n - \sum_{k\in\mathcal{V}_i}\int_{a_{ik}}(\vec{\varphi}_{ij}\times\frac{\vec{\mathbf{E}}_i^n+\vec{\mathbf{E}}_k^n}{2}).\vec{n}_{ik}\right) \\ &= \int_{\mathcal{T}_i}\vec{\text{rot}}\vec{\mathbf{H}}_i^{n\text{-}1/2}.\vec{\mathbf{E}}_i^n - \sum_{k\in\mathcal{V}_i}\int_{a_{ik}}(\vec{\mathbf{H}}_i^{n\text{-}1/2}\times\frac{\vec{\mathbf{E}}_i^n+\vec{\mathbf{E}}_k^n}{2}).\vec{n}_{ik} \\ &= \frac{1}{2}\int_{\mathcal{T}_i}\left(\vec{\text{rot}}\vec{\mathbf{H}}_i^{n\text{-}1/2}.\vec{\mathbf{E}}_i^n + \vec{\text{rot}}\vec{\mathbf{E}}_i^n.\vec{\mathbf{H}}_i^{n\text{-}1/2}\right) - \frac{1}{2}\sum_{k\in\mathcal{V}_i}\int_{a_{ik}}(\vec{\mathbf{H}}_i^{n\text{-}1/2}\times\vec{\mathbf{E}}_k^n).\vec{n}_{ik}.\end{aligned}$$

In the sequel of this proof, we omit the superscripts n and n-1/2 respectively in electric and magnetic variables. We have the following identities:

$$\begin{aligned}|\mathbb{X}_i^n| &\leq \frac{1}{2}\left|\int_{\mathcal{T}_i}\vec{\text{rot}}\vec{\mathbf{H}}_i.\vec{\mathbf{E}}_i\right| + \frac{1}{2}\left|\int_{\mathcal{T}_i}\vec{\text{rot}}\vec{\mathbf{E}}_i.\vec{\mathbf{H}}_i\right| + \frac{1}{2}\sum_{k\in\mathcal{V}_i}\int_{a_{ik}}\frac{1}{\sqrt{\mu_i\epsilon_k}}\|\sqrt{\mu_i}\vec{\mathbf{H}}_i\times\sqrt{\epsilon_k}\vec{\mathbf{E}}_k\| \\ &\leq \frac{1}{2}\|\vec{\text{rot}}\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i} + \frac{1}{2}\|\vec{\text{rot}}\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i} + \frac{1}{4}\sum_{k\in\mathcal{V}_i}\left(\sqrt{\frac{\mu_i}{\epsilon_k}}\|\vec{\mathbf{H}}_i\|_{a_{ik}}^2 + \sqrt{\frac{\epsilon_k}{\mu_i}}\|\vec{\mathbf{E}}_k\|_{a_{ik}}^2\right) \\ &\leq \frac{\alpha_i P_i}{V_i}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i} + \frac{1}{4}\sum_{k\in\mathcal{V}_i}\left(\frac{\beta_{ik}\|\vec{n}_{ik}\|\sqrt{\mu_i}}{V_i\sqrt{\epsilon_k}}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 + \frac{\beta_{ki}\|\vec{n}_{ik}\|\sqrt{\epsilon_k}}{V_k\sqrt{\mu_i}}\|\vec{\mathbf{E}}_k\|_{\mathcal{T}_k}^2\right)\end{aligned}$$

This expression is also valid when the considered finite volume has a metallic boundary face, because of the convention in the lemma and because of (11). To conclude the proof, the lower bounds in Definition 3.2 yield the result given in the lemma.

Lemma 3.4 *Using the scheme (9)-(10)-(11), under assumptions of Definitions 3.2 and 3.3, the total discrete electromagnetic energy $\mathbb{E}^n$ defined in Definition 3.1 is a positive definite quadratic form of all unknowns if*

$$\forall i, \forall k \in \mathcal{V}_i, \ \frac{\Delta t}{\sqrt{\epsilon_i \mu_i}} \left[2\alpha_i + \beta_{ik} \max\left(\sqrt{\frac{\mu_i}{\mu_k}}, \sqrt{\frac{\epsilon_i}{\epsilon_k}} \right) \right] < \frac{4V_i}{P_i}.$$

Proof: Following the result of the previous lemma, using the definition of $P_i = \sum_{k \in \mathcal{V}_i} \|\vec{n}_{ik}\|$, we can split $\mathbb{E}_i$ the following way:

$$\begin{aligned} \mathbb{E}_i^n \ \geq \ & \sum_{k \in \mathcal{V}_i} \|\vec{n}_{ik}\| \left(\frac{\epsilon_i}{2P_i} \|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}^2 + \left(\frac{\mu_i}{2P_i} - \frac{\beta_{ik} \Delta t \sqrt{\mu_i}}{8 V_i \sqrt{\epsilon_k}} \right) \|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 \right. \\ & \left. - \frac{\alpha_i \Delta t}{2V_i} \|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i} \|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i} - \frac{\beta_{ki} \Delta t \sqrt{\epsilon_k}}{8 V_k \sqrt{\mu_i}} \|\vec{\mathbf{E}}_k\|_{\mathcal{T}_k}^2 \right). \end{aligned}$$

At this point, we choose to use an upper bound for the term $\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i} \|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}$ which might lead to sub-optimal lower bounds for the energy (and then to a slightly too severe stability limit for the scheme). Anyway, this stability limit is only sufficient, and not really close to necessary. We use the inequality

$$\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i} \|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i} \leq \frac{\sqrt{\mu_i}}{2\sqrt{\epsilon_i}} \|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 + \frac{\sqrt{\epsilon_i}}{2\sqrt{\mu_i}} \|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}^2.$$

We then can sum up the lower bounds for the $\mathbb{E}_i$ to obtain

$$\mathbb{E}^n \geq \sum_{\text{faces } a_{ik}}^{\text{internal}} \|\vec{n}_{ik}\| W_{ik} + \sum_{\text{faces } a_{ik}}^{\text{metallic boundary}} \|\vec{n}_{ik}\| Z_{ik}, \text{ with} \tag{15}$$

$$\begin{aligned} W_{ik} \ = & \left(\frac{\epsilon_i}{2P_i} - \frac{\beta_{ik} \Delta t \sqrt{\epsilon_i}}{8 V_i \sqrt{\mu_k}} - \frac{\alpha_i \Delta t \sqrt{\epsilon_i}}{4 V_i \sqrt{\mu_i}} \right) \|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}^2 + \left(\frac{\mu_i}{2P_i} - \frac{\beta_{ik} \Delta t \sqrt{\mu_i}}{8 V_i \sqrt{\epsilon_k}} - \frac{\alpha_i \Delta t \sqrt{\mu_i}}{4 V_i \sqrt{\epsilon_i}} \right) \|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 + \\ & \left(\frac{\epsilon_k}{2P_k} - \frac{\beta_{ki} \Delta t \sqrt{\epsilon_k}}{8 \sqrt{\mu_i} V_k} - \frac{\alpha_k \Delta t \sqrt{\epsilon_k}}{4 V_k \sqrt{\mu_k}} \right) \|\vec{\mathbf{E}}_k\|_{\mathcal{T}_k}^2 + \left(\frac{\mu_k}{2P_k} - \frac{\beta_{ki} \Delta t \sqrt{\mu_k}}{8 V_k \sqrt{\epsilon_i}} - \frac{\alpha_k \Delta t \sqrt{\mu_k}}{4 V_k \sqrt{\epsilon_k}} \right) \|\vec{\mathbf{H}}_k\|_{\mathcal{T}_k}^2 \\ Z_{ik} \ = & \left(\frac{\epsilon_i}{2P_i} - \frac{\beta_{ik} \Delta t \sqrt{\epsilon_i}}{8 V_i \sqrt{\mu_i}} - \frac{\alpha_i \Delta t \sqrt{\epsilon_i}}{4 V_i \sqrt{\mu_i}} \right) \|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}^2 + \left(\frac{\mu_i}{2P_i} - \frac{\beta_{ik} \Delta t \sqrt{\mu_i}}{8 V_i \sqrt{\epsilon_i}} - \frac{\alpha_i \Delta t \sqrt{\mu_i}}{4 V_i \sqrt{\epsilon_i}} \right) \|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 \end{aligned}$$

Finally, all quadratic forms are positive definite if

$$\forall i, \forall k \in \mathcal{V}_i, \ \begin{cases} \dfrac{\epsilon_i}{2P_i} - \dfrac{\beta_{ik} \Delta t \sqrt{\epsilon_i}}{8 V_i \sqrt{\mu_k}} - \dfrac{\alpha_i \Delta t \sqrt{\epsilon_i}}{4 V_i \sqrt{\mu_i}} > 0, \\ \dfrac{\mu_i}{2P_i} - \dfrac{\beta_{ik} \Delta t \sqrt{\mu_i}}{8 V_i \sqrt{\epsilon_k}} - \dfrac{\alpha_i \Delta t \sqrt{\mu_i}}{4 V_i \sqrt{\epsilon_i}} > 0, \end{cases}$$

which is equivalent to the condition given in the lemma. Under that condition, all quadratic forms are positive definite, and the energy $\mathbb{E}^n$ as well. The reader can check that this condition also includes the treatment of metallic boundaries. This result leads the main result of this section:

THEOREM 3.1 *Using the scheme (9)-(10)-(11) on arbitrary finite volumes as described in this section (with metallic boundary conditions only), under assumptions of Definitions 3.2 and 3.3, the energy $\mathbb{E}^n$ defined in Definition 3.1 is conserved through iterations. It is also a positive definite quadratic form of all unknowns $(\mathbf{E}_i^n, \mathbf{H}_i^{n-1/2})$, and therefore the scheme is L^2-stable, if the time step Δt is such that*

$$\forall i, \forall k \in \mathcal{V}_i, \ \frac{\Delta t}{\sqrt{\epsilon_i \mu_i}} \left[2\alpha_i + \beta_{ik} \max\left(\sqrt{\frac{\mu_i}{\mu_k}}, \sqrt{\frac{\epsilon_i}{\epsilon_k}} \right) \right] < \frac{4V_i}{P_i}$$

(with the convention that k should be replaced by i in the above formula for metallic boundary interfaces a_{ik}).

REMARK 3.1 *CFL nature of this sufficient stability condition.*

The above stability condition is CFL-type, as the parameters α_i and β_{ik} are dimensionless, the fraction V_i/P_i has the dimension of a length and gives an approximate for the local size of the finite volume, and finally, $1/\sqrt{\epsilon_i \mu_i}$ has the dimension of a wave speed (it is indeed an upper bound for the local wave speed in the heterogeneous anisotropic medium).

4 A stable scheme with absorbing boundary conditions

In this section, we deal with absorbing boundary conditions. We aim at proposing some weak treatment for an absorbing boundary and proving a sufficient condition for the L^2-stability of the new Galerkin-Discontinuous scheme (9-10) with both metallic and absorbing boundaries. We use again the same energy approach as previously: we show that some discrete energy, playing the role of a Lyapunov function of the whole set of numerical unknowns under some positivity condition, is non-increasing.

4.1 Weak treatment of absorbing boundaries

In the following, a first-order Silver–Müller absorbing condition is used on the absorbing boundary $\partial\Omega_a$. We recall we have assumed that the medium is isotropic near the absorbing boundary $\partial\Omega_a$ (then the permeability and permittivity tensors $\bar{\bar{\varepsilon}}$ and $\bar{\bar{\mu}}$ are scalars). Using the wave speed $c = 1/\sqrt{\mu\epsilon}$, the Silver–Müller absorbing condition can be written

$$\vec{\tilde{n}} \times E = -c\mu \ \vec{\tilde{n}} \times \left(\vec{\tilde{n}} \times H \right), \qquad \vec{\tilde{n}} \times H = c\epsilon \ \vec{\tilde{n}} \times \left(\vec{\tilde{n}} \times E \right),$$

where $\vec{n}$ is the outgoing unitary normal. This boundary condition is exact for outgoing plane waves (with a wave vector collinear with $\vec{n}$). This condition is a first-order approximation, asymptotically correct when the fictitious absorbing boundary is far enough.

In view of the absorbing boundary condition above, we propose the following fictitious fields $\vec{\mathbf{H}}_k^{n+1/2}$ and $\vec{\mathbf{E}}_k^{n+1}$ over an absorbing interface a_{ik} between the real finite volume $\mathcal{T}_i$ and its fictitious neighbour $\mathcal{T}_k$ (these fields are used in the scheme (9)-(10) for the absorbing interface a_{ik}):

$$\text{ABSORBING BOUNDARY:} \quad \forall x \in a_{ik}, \ \left\{ \begin{array}{l} \vec{\mathbf{H}}_k^{n+1/2}(x) = c_i \epsilon_i \ \vec{n}_{ik} \times \vec{\mathbf{E}}_i^n(x), \\ \vec{\mathbf{E}}_k^{n+1}(x) = -c_i \mu_i \ \vec{n}_{ik} \times \vec{\mathbf{H}}_i^{n+1/2}(x), \end{array} \right. \tag{16}$$

where $c_i = 1/\sqrt{\mu_i \epsilon_i}$ is the local wave speed (recall the medium is assumed isotropic near the absorbing boundary).

REMARK 4.1 *Validity and origin of these absorbing boundary fields.*

- these definitions only on a_{ik} of the fictitious fields are sufficient in view of the scheme written as in (9)-(10) (after time discretization);
- these definitions do not have the same form as the absorbing boundary conditions above. Anyway, the closer form $\vec{n}_{ik} \times \vec{\mathbf{H}}_k^{n+1/2}(x) = c_i \epsilon_i \ \vec{n}_{ik} \times \Big(\vec{n}_{ik} \times \vec{\mathbf{E}}_i^n(x)\Big)$ (for instance) is equivalent because fictitious fields are always cross-multiplied by the local normal;
- the reader can check that the fields $\vec{\mathbf{E}}_i^n(x)$ and $\vec{\mathbf{H}}_i^{n+1/2}(x)$ are available when the boundary fluxes are needed to advance them in time. One can also notice that the proposed formulae are time-inconsistent. They probably lead to only first-order accurate absorbing boundary conditions.
- among many possible choices, the origin of these fluxes is not really obvious. In fact, these values correspond to upwind fluxes at the absorbing boundary, based on the hyperbolic nature of the global six-component Maxwell system.

If metallic and absorbing boundary conditions are used, then we can give the variation of the total electromagnetic energy given in Definition (3.1). This is the result of the following lemma.

LEMMA 4.1 *Using the scheme (9)-(10)-(11)-(16), under assumptions of Definitions 3.2 and 3.3, and assuming the material is isotropic near absorbing boundaries, the variation of the discrete electromagnetic energy defined in Definition 3.1 through a time-step is given by*

$$\mathbb{E}^{n+1} = \mathbb{E}^n - \frac{\Delta t}{2} \sum_{faces \ a_{ik}}^{absorbing} \int_{a_{ik}} \left[\begin{array}{r} c_i \mu_i \Big(\vec{n}_{ik} \times \vec{\mathbf{H}}_i^{n+1/2}\Big) \cdot \left(\vec{n}_{ik} \times \dfrac{\vec{\mathbf{H}}_i^{n-1/2} + \vec{\mathbf{H}}_i^{n+1/2}}{2}\right) \\ + c_i \epsilon_i \Big(\vec{n}_{ik} \times \vec{\mathbf{E}}_i^n\Big) \cdot \left(\vec{n}_{ik} \times \dfrac{\vec{\mathbf{E}}_i^n + \vec{\mathbf{E}}_i^{n+1}}{2}\right) \end{array} \right].$$

Proof: This is a direct consequence of Lemma 3.1. We just have to use the definitions of fictitious fields near absorbing boundaries given in (16) and the result of the lemma is found. We can point out here that the discrete energy is not anymore conserved (this is natural since we want waves to go out). It is probably non-increasing for very small time steps, but some additional work has to be done to prove it is non-increasing for all time steps.

4.2 Definition of a non-increasing corrected discrete energy

We propose to correct the discrete energy $\mathbb{E}^n$ proposed in Definition (3.1). Let us then introduce the local and global corrected discrete energies $\mathbb{F}_i^n$ and $\mathbb{F}^n$.

Definition 4.1 *For a complete polyhedral finite volume partition $\mathcal{G}$ of the domain Ω with metallic or absorbing boundary conditions, we define the following corrected electromagnetic energies:*

$$(i)\ \forall i,\ \mathbb{F}_i^n = \frac{1}{2}\int_{\mathcal{T}_i}\left({}^t\vec{\mathbf{E}}_i^n\,\bar{\bar{\varepsilon}}_i\vec{\mathbf{E}}_i^n + {}^t\vec{\mathbf{H}}_i^{n-1/2}\bar{\bar{\mu}}_i\vec{\mathbf{H}}_i^{n+1/2}\right) + \frac{\Delta t}{8}\sum_{faces\ a_{ik}}^{absorbing}\int_{a_{ik}}\left(c_i\mu_i\left\|\vec{\tilde{n}}_{ik}\times\vec{\mathbf{H}}_i^{n-1/2}\right\|^2 - c_i\epsilon_i\left\|\vec{\tilde{n}}_{ik}\times\vec{\mathbf{E}}_i^n\right\|^2\right)$$

$$(ii)\ \mathbb{F}^n = \sum_{i\in\mathcal{G}}\mathbb{F}_i^n.$$

The physical meaning of this corrected discrete energies is not straightforward. Correction terms are only related to absorbing boundaries (which means that $\mathbb{F}^n = \mathbb{E}^n$ if there are none). The additional terms probably find their origin in the temporal inconsistency of boundary numerical fluxes. We can now prove that the discrete energy $\mathbb{F}^n$ is non-increasing.

Lemma 4.2 *Using the scheme (9)-(10)-(11)-(16), under assumptions of Definitions 3.2 and 3.3, and assuming the material is isotropic near absorbing boundaries, the corrected discrete energy $\mathbb{F}^n$ defined in Definition 4.1 is non-increasing. More precisely, the variation $\Delta\mathbb{F} = \mathbb{F}^{n+1} - \mathbb{F}^n$ is given by*

$$\Delta\mathbb{F} = -\frac{\Delta t}{2}\sum_{faces\ a_{ik}}^{absorbing}\int_{a_{ik}}\left(c_i\mu_i\left\|\vec{\tilde{n}}_{ik}\times\frac{\vec{\mathbf{H}}_i^{n-1/2}+\vec{\mathbf{H}}_i^{n+1/2}}{2}\right\|^2 + c_i\epsilon_i\left\|\vec{\tilde{n}}_{ik}\times\frac{\vec{\mathbf{E}}_i^n+\vec{\mathbf{E}}_i^{n+1}}{2}\right\|^2\right) \le 0.$$

Proof. The proof is elementary. We simply add terms deriving from $\mathbb{F}^n - \mathbb{E}^n$, $\mathbb{F}^{n+1} - \mathbb{E}^{n+1}$ and from the result of Lemma 4.1. We have

$$\Delta\mathbb{F} = (\mathbb{F}^{n+1} - \mathbb{E}^{n+1}) + (\mathbb{E}^{n+1} - \mathbb{E}^n) + (\mathbb{E}^n - \mathbb{F}^n) = \Delta t\sum_{\text{faces } a_{ik}}^{\text{absorbing}}\int_{a_{ik}}(c_i\mu_i A_{ik} + c_i\epsilon_i B_{ik}),\ \text{with}$$

$$
\begin{aligned}
A_{ik} &= \frac{1}{8}\left\|\vec{\tilde{n}}_{ik}\times\vec{\mathbf{H}}_i^{n+1/2}\right\|^2-\frac{1}{2}\left(\vec{\tilde{n}}_{ik}\times\vec{\mathbf{H}}_i^{n+1/2}\right).\left(\vec{\tilde{n}}_{ik}\times\frac{\vec{\mathbf{H}}_i^{n-1/2}+\vec{\mathbf{H}}_i^{n+1/2}}{2}\right)-\frac{1}{8}\left\|\vec{\tilde{n}}_{ik}\times\vec{\mathbf{H}}_i^{n-1/2}\right\|^2\\
&= -\frac{1}{2}\left\|\vec{\tilde{n}}_{ik}\times\left(\frac{\vec{\mathbf{H}}_i^{n-1/2}+\vec{\mathbf{H}}_i^{n+1/2}}{2}\right)\right\|^2, \text{ and}\\
B_{ik} &= -\frac{1}{8}\left\|\vec{\tilde{n}}_{ik}\times\vec{\mathbf{E}}_i^{n+1}\right\|^2-\frac{1}{2}\left(\vec{\tilde{n}}_{ik}\times\vec{\mathbf{E}}_i^{n}\right).\left(\vec{\tilde{n}}_{ik}\times\frac{\vec{\mathbf{E}}_i^{n}+\vec{\mathbf{E}}_i^{n+1}}{2}\right)+\frac{1}{8}\left\|\vec{\tilde{n}}_{ik}\times\vec{\mathbf{E}}_i^{n}\right\|^2\\
&= -\frac{1}{2}\left\|\vec{\tilde{n}}_{ik}\times\left(\frac{\vec{\mathbf{E}}_i^{n}+\vec{\mathbf{E}}_i^{n+1}}{2}\right)\right\|^2,
\end{aligned}
$$

which simply leads to the result of the lemma.

4.3 Definite positivity of the corrected discrete energy

In order to prove that our scheme is stable when used with both metallic and absorbing boundary conditions, we finally show that the corrected discrete energy $\mathbb{F}^n$, under some stability condition on Δt, is a positive definite quadratic form of the numerical unknowns $H_i^{n-1/2}$ and E_i^n. This will lead to the stability result of this section.

Lemma 4.3 *Using the scheme (9)-(10)-(11)-(16), under assumptions of Definitions 3.2 and 3.3, and assuming the material is isotropic near absorbing boundaries, the local discrete electromagnetic energy $\mathbb{F}_i^n$ defined in Definition 4.1 verifies,*

$$
\begin{aligned}
\mathbb{F}_i^n \;\geq\; & \frac{\epsilon_i}{2}\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}^2+\frac{\mu_i}{2}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2-\frac{\alpha_i P_i\Delta t}{2V_i}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}\\
& -\frac{\Delta t}{8}\sum_{k\in\mathcal{V}_i}\left(\frac{\beta_{ik}\|\vec{n}_{ik}\|}{V_i}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2+\frac{\beta_{ki}\|\vec{n}_{ik}\|}{V_k}\|\vec{\mathbf{E}}_k\|_{\mathcal{T}_k}^2\right).
\end{aligned}
$$

In the above expression, if the face a_{ik} is a boundary face, then we set by convention $\|\vec{\mathbf{E}}_k\|_{\mathcal{T}_k}\equiv\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}$, $\|\vec{\mathbf{H}}_k\|_{\mathcal{T}_k}\equiv\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}$, $\beta_{ki}\equiv\beta_{ik}$, $V_k\equiv V_i$, $\epsilon_k\equiv\epsilon_i$, and $\mu_k\equiv\mu_i$.

Proof: We get back to the definition of the corrected discrete energy inside a finite volume $\mathbb{F}_i^n$. We have

$$\begin{aligned}
\mathbb{F}_i^n &= \frac{1}{2}{}^t\mathbf{E}_i^n M_i^\epsilon \mathbf{E}_i^n + \frac{1}{2}{}^t\mathbf{H}_i^{n-1/2} M_i^\mu \mathbf{H}_i^{n+1/2} \\
&\quad + \frac{\Delta t}{8} \sum_{\text{faces } a_{ik}}^{\text{absorbing}} \int_{a_{ik}} \left(c_i\mu_i \left\| \vec{\tilde{n}}_{ik} \times \vec{\mathbf{H}}_i^{n-1/2} \right\|^2 - c_i\epsilon_i \left\| \vec{\tilde{n}}_{ik} \times \vec{\mathbf{E}}_i^n \right\|^2 \right) \\
&= \frac{1}{2}{}^t\mathbf{E}_i^n M_i^\epsilon \mathbf{E}_i^n + \frac{1}{2}{}^t\mathbf{H}_i^{n-1/2} M_i^\mu \mathbf{H}_i^{n-1/2} - \frac{\Delta t}{2}\mathbb{X}_i^n + \frac{\Delta t}{8}\mathbb{Y}_i^n, \text{ with} \\
\mathbb{Y}_i^n &= \sum_{\text{faces } a_{ik}}^{\text{absorbing}} \int_{a_{ik}} \left(c_i\mu_i \left\| \vec{\tilde{n}}_{ik} \times \vec{\mathbf{H}}_i^{n-1/2} \right\|^2 - c_i\epsilon_i \left\| \vec{\tilde{n}}_{ik} \times \vec{\mathbf{E}}_i^n \right\|^2 \right) \\
\mathbb{X}_i^n &= \sum_{1\le j\le d_i} {}^tH_{ij}^{n-1/2} \left(\int_{\mathcal{T}_i} \vec{\text{rot}}\vec{\varphi}_{ij}.\vec{\mathbf{E}}_i^n - \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} (\vec{\varphi}_{ij} \times \frac{\vec{\mathbf{E}}_i^n + \vec{\mathbf{E}}_k^n}{2}).\vec{\tilde{n}}_{ik} \right) \\
&= \int_{\mathcal{T}_i} \vec{\text{rot}}\vec{\mathbf{H}}_i^{n-1/2}.\vec{\mathbf{E}}_i^n - \sum_{k\in\mathcal{V}_i} \int_{a_{ik}} (\vec{\mathbf{H}}_i^{n-1/2} \times \frac{\vec{\mathbf{E}}_i^n + \vec{\mathbf{E}}_k^n}{2}).\vec{\tilde{n}}_{ik} \\
&= \frac{1}{2}\int_{\mathcal{T}_i} \left(\vec{\text{rot}}\vec{\mathbf{H}}_i^{n-1/2}.\vec{\mathbf{E}}_i^n + \vec{\text{rot}}\vec{\mathbf{E}}_i^n.\vec{\mathbf{H}}_i^{n-1/2} \right) - \frac{1}{2}\sum_{k\in\mathcal{V}_i} \int_{a_{ik}} (\vec{\mathbf{H}}_i^{n-1/2} \times \vec{\mathbf{E}}_k^n).\vec{\tilde{n}}_{ik} \\
&= \frac{1}{2}\int_{\mathcal{T}_i} \left(\vec{\text{rot}}\vec{\mathbf{H}}_i^{n-1/2}.\vec{\mathbf{E}}_i^n + \vec{\text{rot}}\vec{\mathbf{E}}_i^n.\vec{\mathbf{H}}_i^{n-1/2} \right) - \frac{1}{2}\sum_{\text{faces } a_{ik}}^{\text{internal}} \int_{a_{ik}} (\vec{\mathbf{H}}_i^{n-1/2} \times \vec{\mathbf{E}}_k^n).\vec{\tilde{n}}_{ik} \\
&\quad + \frac{1}{2}\sum_{\text{faces } a_{ik}}^{\text{metallic}} \int_{a_{ik}} (\vec{\mathbf{H}}_i^{n-1/2} \times \vec{\mathbf{E}}_i^n).\vec{\tilde{n}}_{ik} - \frac{1}{2}\sum_{\text{faces } a_{ik}}^{\text{absorbing}} \int_{a_{ik}} (\vec{\mathbf{H}}_i^{n-1/2} \times \vec{\mathbf{E}}_k^n).\vec{\tilde{n}}_{ik}.
\end{aligned}$$

Combining terms in $\mathbb{X}_i^n$ and $\mathbb{Y}_i^n$ deriving from absorbing boundary faces, we get

$$\begin{aligned}
\mathbb{F}_i^n &= \frac{1}{2}{}^t\mathbf{E}_i^n M_i^\epsilon \mathbf{E}_i^n + \frac{1}{2}{}^t\mathbf{H}_i^{n-1/2} M_i^\mu \mathbf{H}_i^{n+1/2} - \frac{\Delta t}{4}\int_{\mathcal{T}_i} \left(\vec{\text{rot}}\vec{\mathbf{H}}_i^{n-1/2}.\vec{\mathbf{E}}_i^n + \vec{\text{rot}}\vec{\mathbf{E}}_i^n.\vec{\mathbf{H}}_i^{n-1/2} \right) \\
&\quad + \frac{\Delta t}{4}\sum_{\text{faces } a_{ik}}^{\text{internal}} \int_{a_{ik}} (\vec{\mathbf{H}}_i^{n-1/2} \times \vec{\mathbf{E}}_k^n).\vec{\tilde{n}}_{ik} - \frac{\Delta t}{4}\sum_{\text{faces } a_{ik}}^{\text{metallic}} \int_{a_{ik}} (\vec{\mathbf{H}}_i^{n-1/2} \times \vec{\mathbf{E}}_i^n).\vec{\tilde{n}}_{ik} \\
&\quad - \frac{\Delta t}{8}\sum_{\text{faces } a_{ik}}^{\text{absorbing}} \int_{a_{ik}} \left(c_i\mu_i \left\| \vec{\tilde{n}}_{ik} \times \vec{\mathbf{H}}_i^{n-1/2} \right\|^2 + c_i\epsilon_i \left\| \vec{\tilde{n}}_{ik} \times \vec{\mathbf{E}}_i^n \right\|^2 \right).
\end{aligned}$$

In the sequel of this proof, we omit the superscripts n and n-1/2 respectively in electric and magnetic variables. We have the following lower bound:

$$\begin{aligned}\mathbb{F}_i^n \quad \geq \quad & \frac{\epsilon_i}{2}\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i}^2 + \frac{\mu_i}{2}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 - \frac{\Delta t \alpha_i P_i}{2V_i}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}\|\vec{\mathbf{E}}_i\|_{\mathcal{T}_i} \\ & -\frac{\Delta t}{8}\sum_{k\in\mathcal{V}_i}\left(\frac{\beta_{ik}\|\vec{n}_{ik}\|\sqrt{\mu_i}}{V_i\sqrt{\epsilon_k}}\|\vec{\mathbf{H}}_i\|_{\mathcal{T}_i}^2 + \frac{\beta_{ki}\|\vec{n}_{ik}\|\sqrt{\epsilon_k}}{V_k\sqrt{\mu_i}}\|\vec{\mathbf{E}}_k\|_{\mathcal{T}_k}^2\right).\end{aligned}$$

This expression is valid for all finite volumes, because of the conventions in the lemma and because of the treatments of boundary conditions (11) and (16). This concludes the proof of the lemma.

Lemma 4.4 *Using the scheme (9)-(10)-(11)-(16), under assumptions of Definitions 3.2 and 3.3, and assuming the material is isotropic near absorbing boundaries, the corrected total discrete electromagnetic energy $\mathbb{F}^n$ defined in Definition 4.1 is a positive definite quadratic form of all unknowns if*

$$\forall i, \forall k \in \mathcal{V}_i, \ \frac{\Delta t}{\sqrt{\epsilon_i\mu_i}}\left[2\alpha_i + \beta_{ik}\max\left(\sqrt{\frac{\mu_i}{\mu_k}}, \sqrt{\frac{\epsilon_i}{\epsilon_k}}\right)\right] < \frac{4V_i}{P_i}.$$

Proof: The proof is exactly the same as in the metallic case. Under the condition of the lemma (which is the same as in the metallic case), the corrected energy is positive definite. We get the following stability result.

Theorem 4.1 *Using the scheme (9)-(10)-(11)-(16) on arbitrary finite volumes as described in this section, under assumptions of Definitions 3.2 and 3.3, and assuming the material is isotropic near absorbing boundaries, the corrected energy $\mathbb{F}^n$ defined in (4.1) is non-increasing through iterations. It is also a positive definite quadratic form of all unknowns $(\mathbf{E}_i^n, \mathbf{H}_i^{n\text{-}1/2})$, and therefore the scheme is L^2-stable, if the time step Δt is such that*

$$\forall i, \forall k \in \mathcal{V}_i, \ \frac{\Delta t}{\sqrt{\epsilon_i\mu_i}}\left[2\alpha_i + \beta_{ik}\max\left(\sqrt{\frac{\mu_i}{\mu_k}}, \sqrt{\frac{\epsilon_i}{\epsilon_k}}\right)\right] < \frac{4V_i}{P_i}$$

(with the convention that k should be replaced by i in the above formula for metallic boundary interfaces a_{ik}).

5 Particular cases on tetrahedral meshes

5.1 Classical finite volumes on tetrahedral meshes

For tetrahedral discretizations of the computing domain, a classical finite volume scheme based on a leap-frog time-scheme and centered fluxes has been proposed [3]. This work is a particular case of the more general formulation proposed in this paper, with the following particular choices:

- the finite volumes are the tetrahedra themselves;
- for each field, three basis functions have been chosen, which are the constant vector fields $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$. Therefore, we have $\forall i \in \mathcal{G},\ d_i = 3$;
- the material is heterogeneous, isotropic, with constant electromagnetic parameters inside each finite volume (i.e. $\forall x \in \mathcal{T}_i,\ \bar{\bar{\varepsilon}}_i(x) = \bar{\epsilon}_i \mathbb{I},\ \bar{\bar{\mu}}_i(x) = \bar{\mu}_i \mathbb{I}$);
- metallic and absorbing boundary conditions are enforced in a weak sense, with the same choices for boundary fluxes.

For this particular choices, in the general framework proposed in this paper, the material is indeed isotropic near absorbing boundaries and assumptions of Definitions 3.2 and 3.3 are verified with

- $\forall i \in \mathcal{G},\ \alpha_i = 0$;
- $\forall i \in \mathcal{G},\ \epsilon_i = \bar{\epsilon}_i,\ \mu_i = \bar{\mu}_i$;
- $\forall i \in \mathcal{G}, \forall k \in \mathcal{V}_i,\ \beta_{ik} = 1$;

With these parameters, Theorems 3.1 and 4.1 imply that the finite volume method proposed in [3] has a decreasing discrete corrected total electromagnetic energy, which is a positive definite quadratic form of all unknowns, and therefore the scheme is stable, if

$$\forall i, \forall k \in \mathcal{V}_i,\ \frac{\Delta t}{\sqrt{\epsilon_i \mu_i}} \left[\max \left(\sqrt{\frac{\mu_i}{\mu_k}}, \sqrt{\frac{\epsilon_i}{\epsilon_k}} \right) \right] < \frac{4V_i}{P_i}.$$

This condition is equivalent to

$$\begin{cases} \forall \text{ boundary interface } a_{ik}, & \Delta t < 4\sqrt{\epsilon_i \mu_i} V_i / P_i, \\ \forall \text{ internal interface } a_{ik}, & \Delta t < 4 \min\left(\sqrt{\epsilon_i \mu_k}, \sqrt{\mu_i \epsilon_k}\right) \min\left(V_i/P_i, V_k/P_k\right), \end{cases}$$

which is slightly more severe than the condition obtained in the less general context of [3], which is equivalent to

$$\begin{cases} \forall \text{ boundary interface } a_{ik}, & \Delta t < 4\sqrt{\epsilon_i \mu_i} V_i / P_i, \\ \forall \text{ internal interface } a_{ik}, & \Delta t < 4 \min\left(\sqrt{\epsilon_i \mu_k}, \sqrt{\mu_i \epsilon_k}\right) \sqrt{(V_i/P_i).(V_k/P_k)}. \end{cases}$$

5.2 A $\mathbb{P}_1$-DG FVTD method on tetrahedral meshes

Inside each tetrahedron, the basis vector fields are simply $\mathbb{P}_1$ fields inside the tetrahedron. This leads to twelve degrees of freedom for each field inside each tetrahedron (three components times four $\mathbb{P}_1$ scalar basis functions). Therefore, we have $\forall i \in \mathcal{G},\ d_i = 12$; Different choices can be made concerning the $\mathbb{P}_1$ basis functions inside tetrahedra. We have tried two different implementations where, inside tetrahedron $\mathcal{T}_i$, the $\mathbb{P}_1$ basis vector fields were of the form $\vec{\varphi}_{ij}(x) = \varphi_{ij}(x).\vec{e}_u$ (where $\vec{e}_u$ is $\vec{e}_x$, $\vec{e}_y$, or $\vec{e}_z$), where the φ_{ij} are

- either the $\mathbb{P}_1$ scalar basis functions equal to 0 on a_{ij} and 1 on the other vertex;
- or the $\mathbb{P}_1$ scalar basis function equal to 1 on a_{ij} and -1 on the other vertex.

In both cases, We have chosen to limit a first implementation where the material is homogeneous and isotropic inside each tetrahedron. Because of this simple choice, for both choices of $\mathbb{P}_1$ basis functions considered, exact integrations were performed for volume integrals over tetrahedra and surface integrals over interfaces. The second choice was considered because exact surface integrals at first sight were less CPU-expensive using Gauss quadrature rules based on values on the centers of edges.

Lemma 5.1 *For both choices, assumptions of Definitions 3.2 and 3.3 are verified with the same values (because the span of both families of basis fields are identical), which are:*

(i) $\forall i \in \mathcal{G},\ \alpha_i^2 = \dfrac{20}{9}\dfrac{\max_{j\in\mathcal{V}_i}(\|\vec{n}_{ij}\|)}{P_i}$;

(ii) $\forall i \in \mathcal{G}, \forall k \in \mathcal{V}_i,\ \beta_{ik} = 8/3$;

Proof: Let us consider the standard $\mathbb{P}_1$ scalar basis functions φ_{ij} (equal to 0 on a_{ij} and 1 on the other vertex). We have the following elementary integrals:

$$\int_{\mathcal{T}_i} \varphi_{ij}\varphi_{ij'} = (1+\delta_{jj'})V_i/20,$$

$$\int_{a_i k} \varphi_{ij}\varphi_{ij'} = (1-\delta_{kj})(1-\delta_{kj'})(1+\delta_{jj'})\|\vec{n}_{ik}\|/12.$$

(i) derives from the fact that if a $\mathbb{P}_1$ vector field writes $\vec{\mathbf{X}} = \sum_{j\in\mathcal{V}_i} \vec{X}_{ij}\ \varphi_{ij}$, then we have $\|\vec{\mathbf{X}}\|^2_{\mathcal{T}_i} \geq \dfrac{V_i}{20}\displaystyle\sum_{j\in\mathcal{V}_i} \|\vec{X}_{ij}\|^2$ because of the first equation above. At the same time,

$$\begin{aligned}
\|\vec{\mathrm{rot}}\vec{\mathbf{X}}\|^2_{\mathcal{T}_i} &= \frac{1}{9V_i}\left(\sum_{j\in\mathcal{V}_i} \vec{n}_{ij}\times\vec{X}_{ij}\right)^2 = \frac{P_i^2}{9V_i}\left(\sum_{j\in\mathcal{V}_i} \frac{\|\vec{n}_{ij}\|}{P_i}\ \vec{\tilde{n}}_{ij}\times\vec{X}_{ij}\right)^2 \\
&\leq \frac{P_i^2}{9V_i}\sum_{j\in\mathcal{V}_i}\frac{\|\vec{n}_{ij}\|}{P_i}\left(\vec{\tilde{n}}_{ij}\times\vec{X}_{ij}\right)^2 \leq \frac{P_i}{9V_i}\max_{j\in\mathcal{V}_i}(\|\vec{n}_{ij}\|)\sum_{j\in\mathcal{V}_i}\|\vec{X}_{ij}\|^2
\end{aligned}$$

Then the first assumption of Definitions 3.3 is verified for α_i given in the lemma.
(ii) derives from the fact that β_{ik} should be the smallest constant such that

$$\begin{pmatrix} 0&0&0&0\\ 0&2&1&1\\ 0&1&2&1\\ 0&1&1&2 \end{pmatrix} \leq \frac{3\beta_{ik}}{5}\begin{pmatrix} 2&1&1&1\\ 1&2&1&1\\ 1&1&2&1\\ 1&1&1&2 \end{pmatrix},$$

which leads to $\beta_{ik} = 8/3$.

These results lead to the following sufficient stability condition for our $\mathbb{P}_1$ Discontinuous Galerkin FVTD method on unstructured tetrahedral meshes:

$$\forall i, \forall j \in \mathcal{V}_i,\ \frac{\Delta t}{\sqrt{\epsilon_i \mu_i}} \left[\frac{4\sqrt{5}}{3} \sqrt{\frac{\max_{j \in \mathcal{V}_i}(\|\vec{n}_{ij}\|)}{P_i}} + \frac{8}{3} \max\left(\sqrt{\frac{\mu_i}{\mu_j}}, \sqrt{\frac{\epsilon_i}{\epsilon_j}} \right) \right] < \frac{4V_i}{P_i}.$$

Remark 5.1 *A simplified version for heterogeneous media.*

It is easy to see that $\max_{j \in \mathcal{V}_i}(\|\vec{n}_{ij}\|) \leq P_i/2$ because if $\|\vec{n}_{ij_0}\| = \max_{j \in \mathcal{V}_i}(\|\vec{n}_{ij}\|)$, then

$$P_i - \|\vec{n}_{ij_0}\| = \sum_{j \in \mathcal{V}_i, j \neq j_0} \|\vec{n}_{ij}\| \geq \left\| \sum_{j \in \mathcal{V}_i, j \neq j_0} \vec{n}_{ij} \right\| = \| - \vec{n}_{ij_0}\| = \|\vec{n}_{ij_0}\|.$$

Then the stability limit can be simplified into

$$\forall i, \forall j \in \mathcal{V}_i,\ \frac{\Delta t}{\sqrt{\epsilon_i \mu_i}} \left[\frac{4\sqrt{5}}{3\sqrt{2}} + \frac{8}{3} \max\left(\sqrt{\frac{\mu_i}{\mu_j}}, \sqrt{\frac{\epsilon_i}{\epsilon_j}} \right) \right] < \frac{4V_i}{P_i}.$$

Remark 5.2 *Comparison with classical finite volumes for homogeneous media.*

For homogeneous media, the above stability condition reduces to

$$\forall i, \forall j \in \mathcal{V}_i,\ \frac{\Delta t}{\sqrt{\epsilon_i \mu_i}} \left[\frac{4\sqrt{5}}{3\sqrt{2}} + \frac{8}{3} \right] < \frac{4V_i}{P_i}.$$

This means that, theoretically, the limit possible time step for our $\mathbb{P}_1$ Discontinuous Galerkin FVTD method on unstructured tetrahedral meshes will be smaller than the limit time step admissible with classical finite volumes by a factor of $\frac{4\sqrt{5}}{3\sqrt{2}} + \frac{8}{3}$, roughly equal to 4.7. We can notice that if tetrahedra are assumed quite equilateral, then $\max_{j \in \mathcal{V}_i}(\|\vec{n}_{ij}\|) \simeq P_i/4$ and the reduction factor is theoretically close to 4.1.

6 Numerical results

We select two resonant cavities, a cubic one and a spherical one, since the exact solutions are known for these geometries allowing us to appreciate the numerical results at any point and time in the cavity.

6.1 The cubic cavity

We compute the $(1,1,1)$ mode which is a standing wave of 0.260 GHz frequency in a cube of 1 m of side. We use an unstructured grid of 16464 tetrahedra and 3375 nodes which gives 13 points per wavelength. We plot on Figure 2 the time evolution at a fixed point in the cavity of the z-component of the electric field during twelve periods. One can see that the Discontinuous Galerkin ($\mathbb{P}_1$-DG) solution compares well with the exact one. The Figure 3

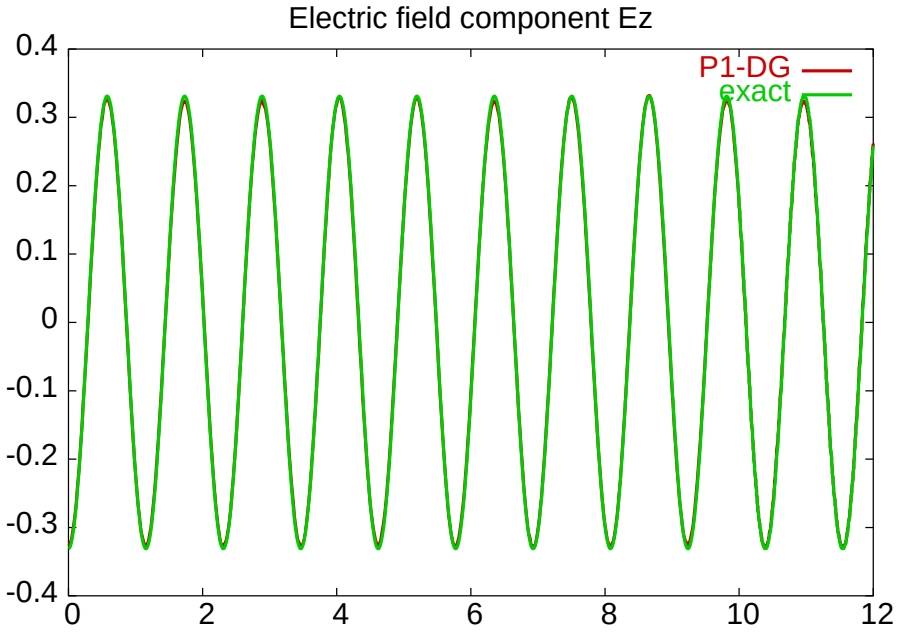

Figure 2: $\mathbb{P}_1$-DG, $\mathbb{P}_0$, and exact solutions: first twelve periods (component E_z).

shows a zoom on the last two periods of the $\mathbb{P}_1$-DG and exact fields with now the $\mathbb{P}_0$ (finite volume) solution which obviously has a higher rate of dispersion error. Let us recall that the finite volume scheme has the same order of dispersion error as the Yee scheme [5]. The overall L^2-error on the electromagnetic field (E,H) of the $\mathbb{P}_1$-DG and $\mathbb{P}_0$ approximate solutions are plotted on Figure 4. The errors are increasing in time because of the dispersion and the level of dispersion is a lot smaller for the $\mathbb{P}_1$-DG method. Figure 5 shows the contours of the E_x and H_z components for the exact and $\mathbb{P}_1$-DG solutions in the cut plane $z=0$. Figure 6 and Figure 7 show the contours respectively of the fields (E_x,E_y,E_z) and (H_x,H_y,H_z) for the exact and $\mathbb{P}_1$-DG solutions in the cut plane $x+y+z=1.5$.

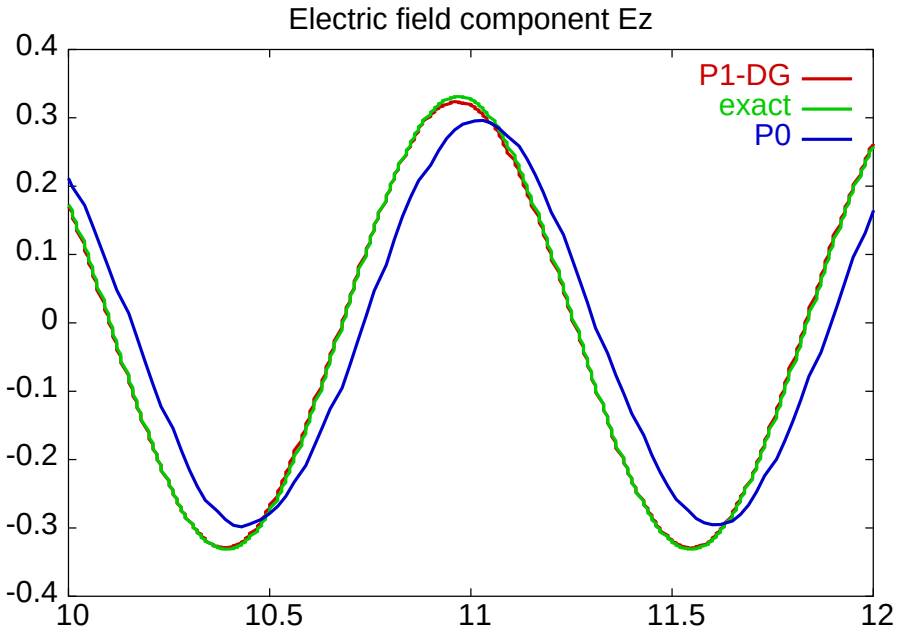

Figure 3: $\mathbb{P}_1$-DG, $\mathbb{P}_0$, and exact solutions: zoom after ten periods (component E_z).

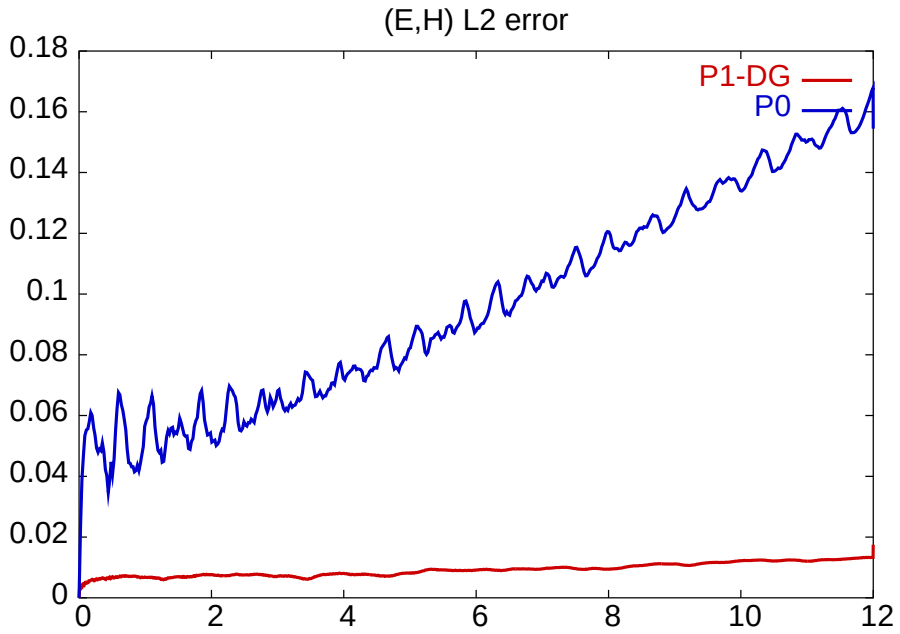

Figure 4: L^2-error on (E, H) for $\mathbb{P}_1$-DG and $\mathbb{P}_0$ approximate solutions in function of the time.

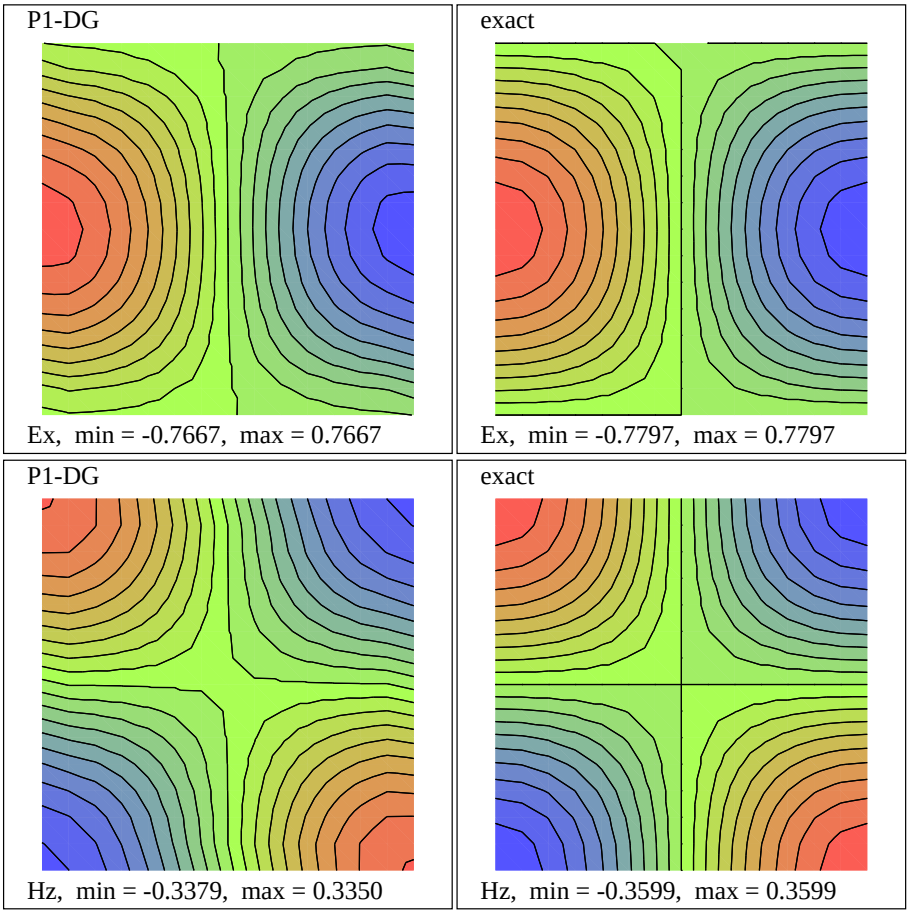

Figure 5: $\mathbb{P}_1$-DG and exact solutions: E_x and H_z contours in plane $z = 0$.

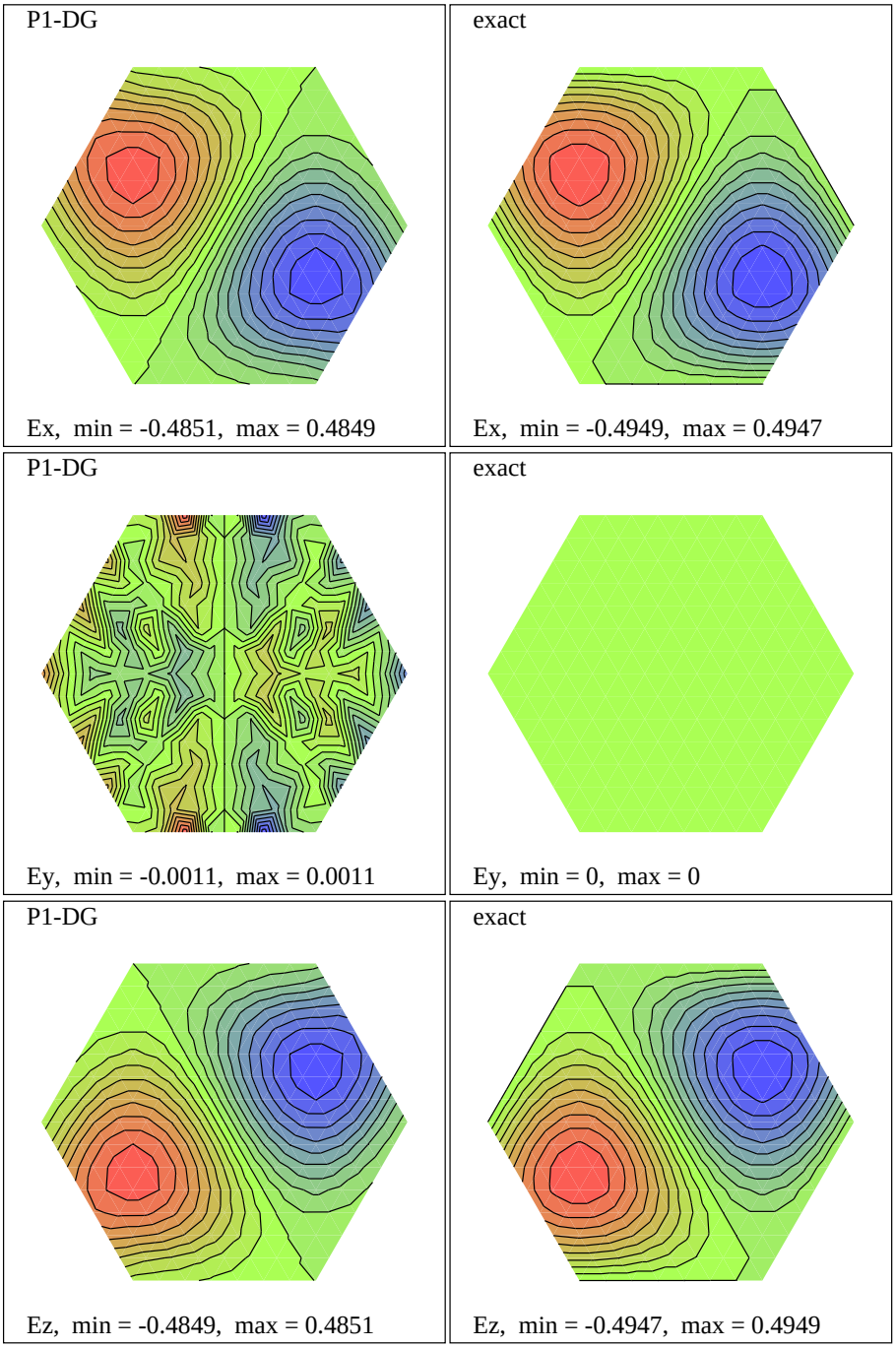

Figure 6: $\mathbb{P}_1$-DG and exact solutions: (E_x, E_y, E_z) contours in plane $x + y + z = 1.5$.

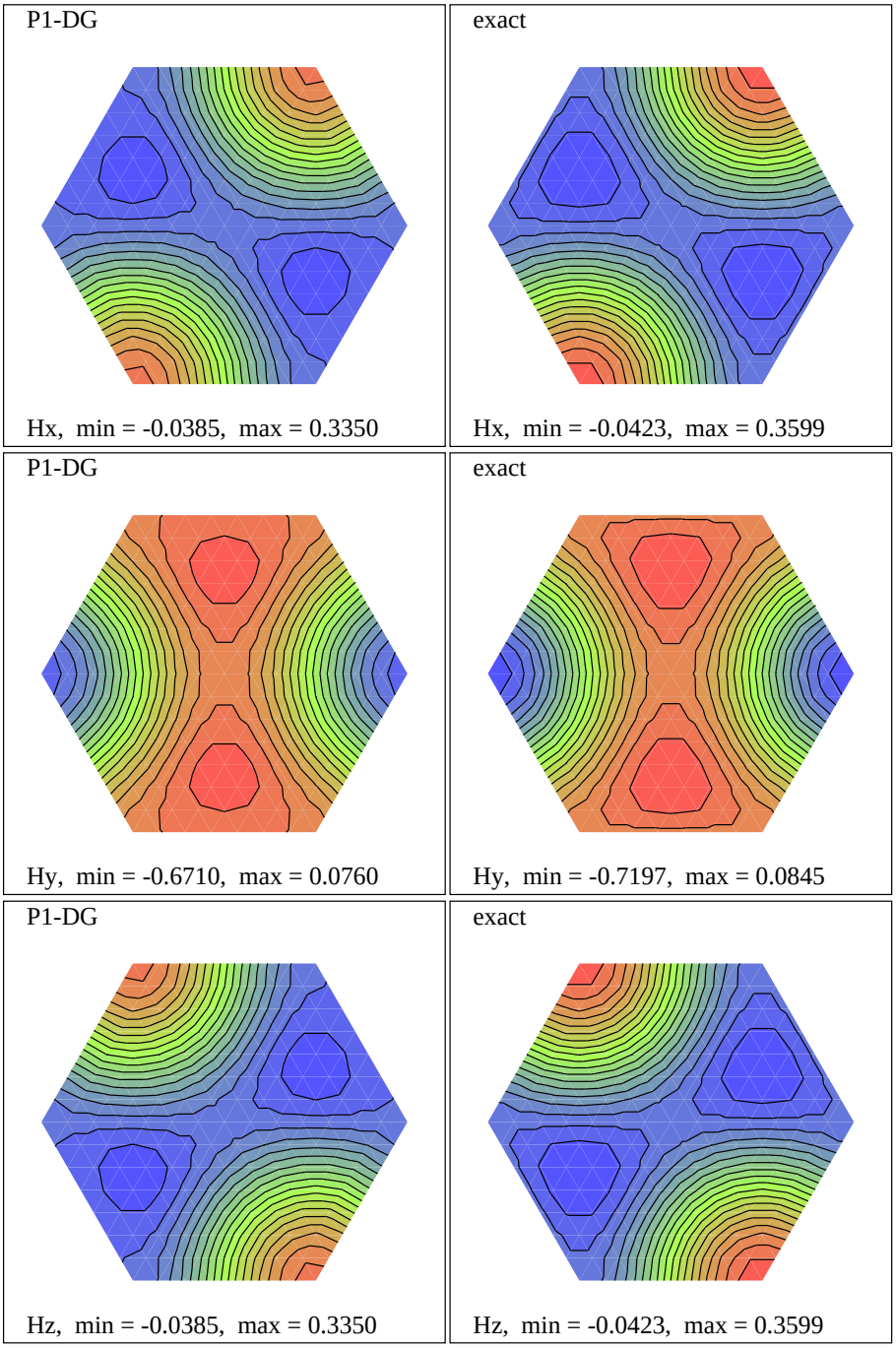

Figure 7: $\mathbb{P}_1$-DG and exact solutions: (H_x, H_y, H_z) contours in plane $x + y + z = 1.5$.

6.2 The spherical cavity

We choose here to compute the lowest $(0, 1, 1)$ TE mode in a spherical cavity of radius 1 m. The resonant frequency is 0.21 GHz and the mesh is made of 82000 tetrahedra and 15000 nodes which corresponds to an average of 12 points per wavelength. We compare on Figure 8 the time evolution of the H_z component of the exact and computed magnetic field during seven periods. One may see again that the two solutions compare very well. These

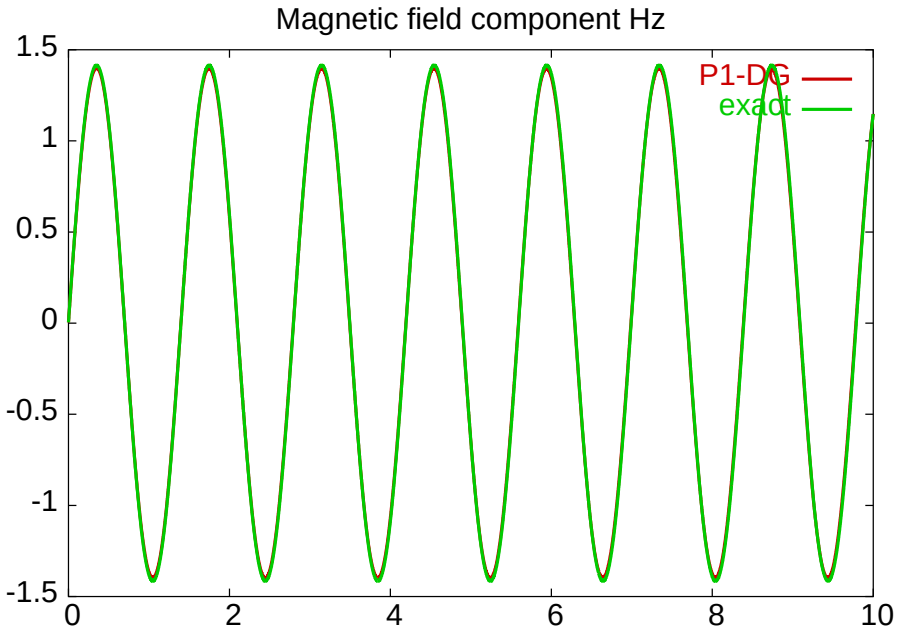

Figure 8: $\mathbb{P}_1$-DG approximate solution vs. exact solution: first seven periods (component H_z).

solutions are compared to the $\mathbb{P}_0$ approximate solution on Figure 9 and the gain in accuracy in favour of the $\mathbb{P}_1$-DG solution is obvious. Figure 10 shows contours of the computed and exact magnetic field respectively in the plane $z = 0$.

7 Conclusion

We presented a new formulation of a $\mathbb{P}_1$ Discontinuous Galerkin method applied to the time domain Maxwell's equations. One may say that it is a simplified formulation when it is compared with the methods found in the literature (see [1] for example). The new method may also be viewed as a straightforward extension of a centered finite volume scheme as the one introduced in [5]. We proved that the method is stable under a CFL like condition

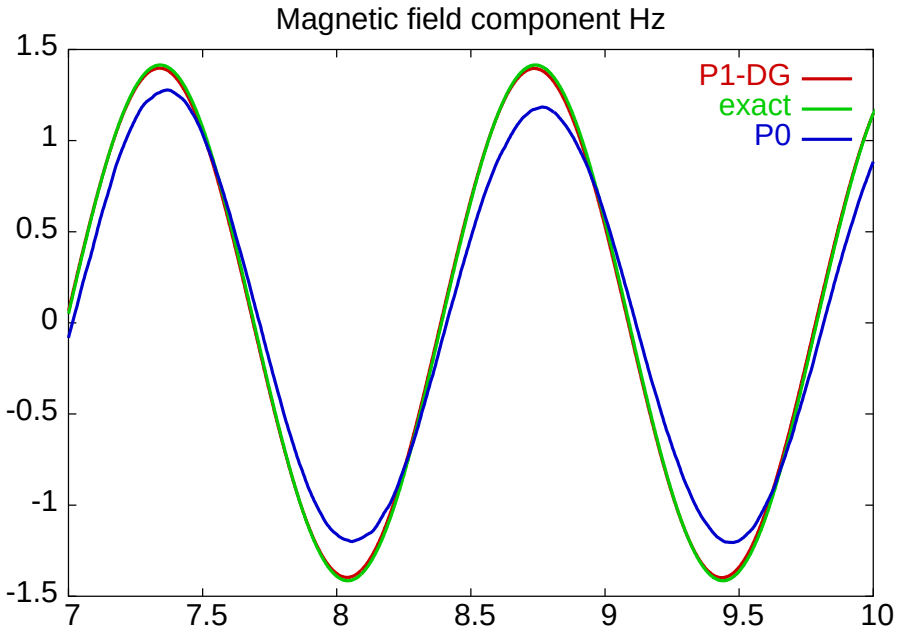

Figure 9: $\mathbb{P}_1$-DG, $\mathbb{P}_0$, and exact solutions: zoom after five periods (component H_z).

and also that a discrete energy is conserved. Some numerical simulations were performed and the results compared with the analytic solutions. One can notice that the numerical results are encouraging, because good accuracy is obtained with few points per wavelength. Higher order of accuracy in time (possibility of a fourth-order accurate scheme conserving an energy) and in space (why not try P2 basis functions) might lead to interesting results. However, due to the limitation on the time-step and the high number of degrees of freedom (24 times the number of cells for tetrahedra and $\mathbb{P}_1$ DG), the method may be considered as very costly in time an memory when compared to the finite volume scheme for example but fortunately the method is highly parallelizable and we expect a very good efficiency. At the same time, the DG methods are very flexible, since the functional basis is local in each finite volume or element. One can imagine to restrict the $\mathbb{P}_1$-DG method to some local zones with complex isolines and use a $\mathbb{P}_0$ method or even a Yee scheme in the vacuum for example. Another possibility would be to adaptively restrict some degrees of freedom in particular sub-domains.

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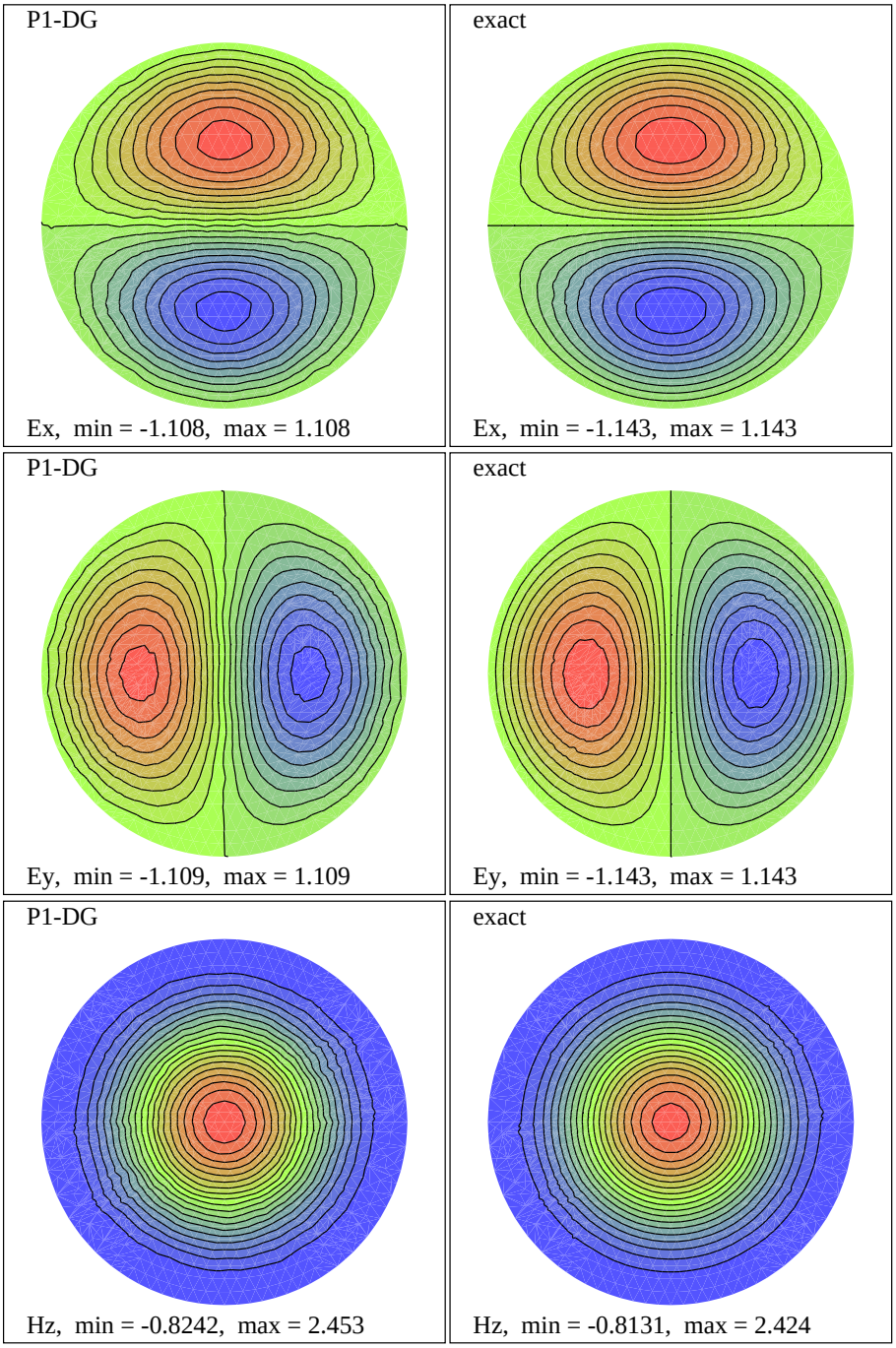

Figure 10: $\mathbb{P}_1$-DG and exact solutions: E_x, E_y, and H_z contours in plane $z = 0$.

Unité de recherche INRIA Sophia Antipolis
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615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
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